7.1

(5) $u'' + 0.25u' + 4u = 2\cos(3t)$, $u(0)=1$, $u'(0)=-2$.

Let $u = x_1$ and $u' = x_2$

Then $x_2' + 0.25x_2 + 4x_1 = 2\cos(3t)$ so $x_2' = -4x_1 - 0.25x_2 + 2\cos(3t)$
and $x_1' = x_2$ $x_1' = x_2$

where $x_1(0) = 1$ and $x_2(0) = -2$.

(9) $x_1' = 1.25x_1 + 0.75x_2$ $x_1(0) = -2$
$x_2' = 0.75x_1 + 1.25x_2$ $x_2(0) = 1$

a) $x_2 = \dfrac{x_1' - 1.25x_1}{0.75} = \frac{4}{3}x_1' - \frac{5}{3}x_1$

so $x_2' = \frac{4}{3}x_1'' - \frac{5}{3}x_1'$ but $x_2' = \frac{3}{4}x_1 + \frac{5}{4}x_2$

$= \frac{3}{4}x_1 + \left(\frac{5}{4}\right)\left(\frac{4}{3}x_1' - \frac{5}{3}x_1\right)$

$= \frac{3}{4}x_1 + \frac{5}{3}x_1' - \frac{25}{12}x_1$

$= \frac{-16}{12}x_1 + \frac{5}{3}x_1' = \frac{5}{3}x_1' - \frac{4}{3}x_1$

so $\frac{4}{3}x_1'' - \frac{5}{3}x_1' = \frac{5}{3}x_1' - \frac{4}{3}x_1$

so $\frac{4}{3}x_1'' - \frac{10}{3}x_1' + \frac{4}{3}x_1 = 0$ so $4x_1'' - 10x_1' + 4x_1 = 0$
so $2x_1'' - 5x_1' + 2x_1 = 0$

b) $2y'' - 5y' + 2y = 0$ so $2r^2 - 5r + 2 = 0$ so $r = \dfrac{5 \pm \sqrt{25-16}}{4} = \dfrac{5 \pm 3}{4} \to 2, +\frac{1}{2}$

so $y(t) = c_1 e^{2t} + c_2 e^{t/2}$.

$y(0) = -2$ because $y = x_1$ $y' = x_1' = \frac{5}{4}x_1 + \frac{3}{4}x_2$ so $y'(0) = \frac{5}{4}(-2) + \frac{3}{4}(1)$
$= \boxed{-7/4}$

So $c_1 + c_2 = -2$ and

$2c_1 + \frac{c_2}{2} = -7/4$

so $2c_1 + 2c_2 = -4$

$2c_1 + \frac{c_2}{2} = -7/4$

so $\frac{3}{2} c_2 = -\frac{9}{4}$ so $c_2 = -3/2$

$c_1 = -2 + 3/2 = -1/2$

so $x_1(t) = -\frac{1}{2} e^{2t} - \frac{3}{2} e^{t/2}$

Then $x_1'(t) = -e^{2t} - \frac{3}{4} e^{t/2}$

But $x_1'(t) = \frac{5}{4} x_1(t) + \frac{3}{4} x_2(t)$, so

$$x_2(t) = \frac{4x_1'(t) - 5x_1(t)}{3}$$

$$x_2(t) = \frac{-4e^{2t} - 3e^{t/2} + \frac{5}{2}e^{2t} + \frac{15}{2}e^{t/2}}{3}$$

$$= \frac{\frac{-8+5}{2} e^{2t} + \frac{15}{2} - 6\, e^{t/2}}{3}$$

$$= -\frac{1}{2} e^{2t} + \frac{3}{2} e^{t/2}$$

so $\boxed{x_2(t) = -\frac{1}{2} e^{2t} + \frac{3}{2} e^{t/2}}$

$\boxed{7.2}$

6c $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$,

$C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$

$$A(B+C) = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 4 & 2 & -1 \\ -1 & 5 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+2 & 2-10 & -1-10 \\ 12-2-1 & 6+10-1 & -3+10-1 \\ -8+3 & -4+3 & 2+3 \end{pmatrix} = \boxed{\begin{pmatrix} 6 & -8 & -11 \\ 9 & 15 & 6 \\ -5 & -1 & 5 \end{pmatrix}}$$

$$AB + AC = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4 & 1-6 & -1-6 \\ 6-4-1 & 3+6 & -3+6-2 \\ -4+3 & -2 & 2+6 \end{pmatrix} + \begin{pmatrix} 2-2 & 1-4 & -4 \\ 6+2 & 3+4-1 & 4+1 \\ -4 & -2+3 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix} + \begin{pmatrix} 0 & -3 & -4 \\ 8 & 6 & 5 \\ -4 & 1 & -3 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 6 & -8 & -11 \\ 9 & 15 & 6 \\ -5 & -1 & 5 \end{pmatrix}}$$

They are equal.

(15)

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/8 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

So $\boxed{\begin{pmatrix} 1/2 & -1/2 & 1/8 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix}}$ is the inverse

$\boxed{7.3}$

(3) $$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $x_1 + x_3 = 0$ $\qquad$ $x_1 = -x_3$

$x_2 - x_3 = 1$ $\qquad$ $x_2 = x_3 + 1$

There are infinitely many solutions. For each value of $x_3$, $x_1 = -x_3$ and $x_2 = x_3 + 1$ is a solution to the system of eqns.

(8) $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is a singular matrix, so the vectors are <u>linearly dependent</u>.

(19) $$\left|\begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix}\right| = (1-\lambda)^2 + i^2 = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda.$$

So $\lambda = 0$ and $\lambda = 2$ are the 2 eigenvalues

If $\lambda = 0$ then $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \vec{x} = 0$ so if $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

then $x_1 + ix_2 = 0$ and $-ix_1 + x_2 = 0$.

$x_1 = 1$ means $x_2 = i$. So $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector of $\lambda = 0$.

(any vector of the form $\begin{pmatrix} k \\ ki \end{pmatrix}$ is an eigenvector of $\lambda = 0$).

For $\lambda = 2$ we have $\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \vec{x} = \vec{0}$

So $-x_1 + ix_2 = 0$ so $x_1 = \frac{x_2}{i}$ i. If $x_2 = 1 \Rightarrow x_1 = i$

$-ix_1 - x_2 = 0$

Then $\begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 2$. (Any multiple of it too).

(25) $\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & (3-\lambda) \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & (3-\lambda) \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix}$

$= (3-\lambda)(-\lambda)(3-\lambda) - 4(3-\lambda) - 2(2)(3-\lambda) + 2(2)(4) + 4(4) - 4(4)(-\lambda)$

$= -(3-\lambda)(3\lambda - \lambda^2 + 4 + 4) + 32 + 16\lambda$

$= (3-\lambda)(\lambda^2 - 3\lambda - 8) + 32 + 16\lambda$

$= -\lambda^3 + 3\lambda^2 + 8\lambda + 3\lambda^2 - 9\lambda - 24 + 32 + 16\lambda$

$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$

If $\lambda = -1$ then $-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 1 + 6 - 15 + 8 = 0$

$$\begin{array}{r} -\lambda^2 + 7\lambda + 8 \\ \lambda+1 \enclose{longdiv}{-\lambda^3 + 6\lambda^2 + 15\lambda + 8} \\ +\lambda^3 + \lambda^2 \\ \hline 7\lambda^2 + 15\lambda \\ -7\lambda^2 - 7\lambda \\ \hline 8\lambda + 8 \\ -8\lambda - 8 \\ \hline 0 \end{array}$$

$-(\lambda+1)(\lambda^2 - 7\lambda - 8)$

$= -(\lambda+1)^2(\lambda - 8)$

So $\lambda = -1$ has algebraic multiplicity 2 and $\lambda = 8$ is the other eigenvalue.

If $\lambda = -1$ then we have $\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \vec{x} = 0$

So $4x_1 + 2x_2 + 4x_3 = 0 \Rightarrow x_1 = -\frac{x_2}{2} - x_3$

$2x_1 + x_2 + 2x_3 = 0$

Let $x_2 = 2$ and $x_3 = 0$, then $x_1 = -1$.
Let $x_2 = 0$ and $x_3 = 1$ then $x_1 = -1$.

So $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are two lin. indep. eigenvectors of

$\lambda = -1$.

If $\lambda = 8$, then $\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \vec{x} = 0$

so
$$-5x_1 + 2x_2 + 4x_3 = 0$$
$$2x_1 - 8x_2 + 2x_3 = 0$$
$$4x_1 + 2x_2 - 5x_3 = 0$$

$$\left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2/5 & -4/5 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc} 1 & -2/5 & -4/5 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right) \longleftarrow \left(\begin{array}{ccc|c} 1 & -2/5 & -4/5 & 0 \\ 0 & \frac{-36}{5} & \frac{18}{5} & 0 \\ 0 & \frac{18}{5} & \frac{-9}{5} & 0 \end{array}\right)$$

So $X_1 - \frac{2}{5} X_2 = \frac{4}{5} X_3$

$$x_2 = \frac{1}{2} x_3.$$

Let $x_3 = 10 \Rightarrow x_2 = 5$ and $x_1 = 10$, So $\begin{pmatrix} 10 \\ 5 \\ 10 \end{pmatrix}$ is an eigenvector

So $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\lambda = 8$.

(29) $\det A = 0$ ~~and~~ $A\vec{x}^{(0)} = \vec{b}$

Let $\xi$ satisfy $A\xi = 0$. Let $\alpha$ be a constant.

Let $\vec{x} = x^{(0)} + \alpha\xi$

Then $A\vec{x} = A\left(x^{(0)} + \alpha\xi\right) = A\vec{x}^{(0)} + A(\alpha\xi)$

$$= A\vec{x}^{(0)} + \alpha(A\xi)$$

$$= A\vec{x}^{(0)} + 0$$

$$= \vec{b} \quad \blacksquare$$