$\boxed{7.4}$

(5) $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$.

Let $\vec{x}^{(c)}$ be the general solution to $\vec{x}' = P(t)\vec{x}$,

i.e. $\vec{x}^{(c)\prime} = P(t)\vec{x}^{(c)}$.

Let $\vec{x}^{(p)}$ be a particular solution of $P(t)\vec{x} + \vec{g}(t) = \vec{x}'$,

so $\vec{x}^{(p)\prime} = P(t)\vec{x}^{(p)} + \vec{g}(t)$.

Then if $\vec{x} = \vec{x}^{(c)} + \vec{x}^{(p)}$,

$$\vec{x}' = \vec{x}^{(c)\prime} + \vec{x}^{(p)\prime}$$

$$= P(t)\vec{x}^{(c)} + P(t)\vec{x}^{(p)} + \vec{g}(t)$$

$$= P(t)\left(\vec{x}^{(c)} + \vec{x}^{(p)}\right) + \vec{g}(t)$$

$$= P(t)\vec{x} + \vec{g}(t) \quad \boxed{\times}$$

(6)

a) $W\left[\vec{x}^{(1)}, \vec{x}^{(2)}\right] = \det\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2$.

b) As long as $t \neq 0$, $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are linearly independent.

So $(-\infty, 0)$ or $(0, \infty)$ are the biggest intervals, but any interval not containing 0 works.

c) At least one coefficient must be discontinuous at $t = 0$.

$$\vec{x}' = P(t)\,\vec{x} \qquad \vec{x}_1^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{so let } \vec{x} = c_1 \vec{x}_1^{(1)} + c_2 \vec{x}_2^{(2)} \qquad \vec{x}_2^{(2)}(t)' = \begin{pmatrix} 2t \\ 2 \end{pmatrix}$$

Then $\vec{x}' = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + 2c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}$

But $P(t)\,\vec{x} = P(t)\left(c_1 \begin{pmatrix} t \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 2t \end{pmatrix} \right)$

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}$$

so

$$\begin{pmatrix} c_1 + 2c_2 t \\ 0 + 2c_2 \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \begin{pmatrix} c_1 t + c_2 t^2 \\ c_1 + 2c_2 t \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}(t)(c_1 t + c_2 t^2) + p_{12}(t)(c_1 + 2c_2 t) \\ p_{21}(t)(c_1 t + c_2 t^2) + p_{22}(t)(c_1 + 2c_2 t) \end{pmatrix}$$

So $c_1 + 2c_2 t = c_1\big(t p_{11}(t) + p_{12}(t)\big) + c_2\big(t^2 p_{11}(t) + 2p_{12}(t)\,t\big)$

$$2c_2 = \big(p_{21}(t)\,t + p_{22}(t)\big)c_1 + \big(t^2 p_{21}(t) + 2p_{22}(t)\,t\big)c_2$$

$$\begin{matrix} t p_{11}(t) + p_{12}(t) = 1 \\ t^2 p_{11}(t) + 2p_{12}(t) = 2t \end{matrix} \rightarrow \begin{matrix} t^2 p_{11} + t p_{12} = t \\ t^2 p_{11} + 2p_{12} t = 2t \end{matrix} \quad \text{so} \quad t p_{12} = t \quad \text{so } p_{12}(t) = 1$$

Then $p_{11}(t) = 0.$

Also

$$\begin{matrix} t p_{21} + p_{22} = 0 \\ t^2 p_{21} + 2t p_{22} = 2 \end{matrix} \Rightarrow \begin{matrix} t^2 p_{21} + t p_{22} = 0 \\ t^2 p_{21} + 2t p_{22} = 2 \end{matrix} \longrightarrow t p_{22} = 2 \quad \text{so } p_{22}(t) = \frac{2}{t}$$

Then $t p_{21} = -p_{22} = -\frac{2}{t}$, so $p_{21}(t) = -\frac{2}{t^2}$

Therefore $\boxed{\vec{x}' = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix} \vec{x}}$ Indeed $2/t$ and $-2/t^2$ are discontinuous at $t=0$.

7.5: 9, 11, 12, 14

7.5

(9) $x' = \begin{pmatrix} 1 & i \\ -i & i \end{pmatrix} \vec{x}$

Let's find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda^2 - 2\lambda$$

so $\lambda = 0$ and $\lambda = 2$. $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ is a singular matrix.

Let's find the eigenvector for $\lambda = 0$:

$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $x_1 + ix_2 = 0$ so $x_1 = -ix_2$

if $x_1 = 1 \Rightarrow x_2 = \frac{1}{-i} = i$

So $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector for $\lambda = 0$.

Let's find the eigenvector for $\lambda = 2$:

$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $-x_1 + ix_2 = 0$

so $x_1 = ix_2$

so if $x_1 = 1$, $x_2 = -i$.

Then the eigenvector is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

So the general solution is

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2t}$$

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2t}}$$

(So $x_1 = c_1 + c_2 e^{2t}$ while $x_2 = ic_1 - ic_2 e^{2t}$)

(11) $x' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} x.$

Eigenvalues: $\begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(2-\lambda)(1-\lambda) + 2 + 2 - 4(2-\lambda) - (1-\lambda) - (1-\lambda)$$

$$= (1-2\lambda+\lambda^2)(2-\lambda) + 4 - 8 + 4\lambda - 1 + \lambda - 1 + \lambda$$

$$= 2 - 4\lambda + 2\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 4 - 10 + 6\lambda$$

$$= -4 + \lambda + 4\lambda^2 - \lambda^3$$

So $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

So $\lambda = 1$ is an eigenvalue.

Then
$$\begin{array}{r} \lambda^2 - 3\lambda - 4 \\ \lambda - 1 \,\overline{)\, \lambda^3 - 4\lambda^2 - \lambda + 4} \\ -\lambda^3 + \lambda^2 \qquad\qquad \\ \hline -3\lambda^2 - \lambda \qquad \\ +3\lambda^2 - 3\lambda \qquad \\ \hline -4\lambda + 4 \\ \hline 0 \end{array}$$

$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$

So $\boxed{\lambda = -1,\ 1 \text{ and } 4}$

When $\lambda = -1$

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ so } \begin{pmatrix} 2 & 1 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $5x_2 = 0$, so $x_2 = 0$. Then $2x_1 + 2x_3 = 0$

so $x_1 = -x_3$

Therefore $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector when $\lambda = -1$.

When $\lambda = 1$ we have

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $x_2 + 2x_3 = 0$ so $x_2 = -2x_3$ and

and $x_1 + x_2 + x_3 = 0$ $x_1 - 2x_3 + x_3 = 0$ so

$x_1 = x_3$

Then $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 1$.

When $\lambda = 4$ we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -5 & 5 \\ 1 & -2 & 1 \\ 0 & +5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $5x_2 = +5x_3$ so $x_2 = x_3$

$x_1 - 2x_2 + x_3 = 0$

so $x_1 = x_3$

So $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

Then the general solution is

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}}$$

(12) $\dot{x} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} x$, $\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix}$

$= -(3-\lambda)\lambda(3-\lambda) - 4(3-\lambda) - 2(2)(3-\lambda) + 2(2)(4) + 4(2)(2) - 4(4)(-\lambda)$

$= -\lambda(9-6\lambda+\lambda^2) \underline{-12+4\lambda} \underline{-12+4\lambda} + \underline{16} + \underline{16} + 16\lambda$

$= -9\lambda + 6\lambda^2 - \lambda^3 + 8 + 24\lambda$

$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$.

$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$. Then $\lambda = -1$ is a solution

since $-1 - 6 + 15 - 8 = 0$.

Now
$$\begin{array}{r|l} & \lambda^2 - 7\lambda - 8 \\ \lambda+1 & \lambda^3 - 6\lambda^2 - 15\lambda - 8 \\ & -\lambda^3 - \lambda^2 \\ \hline & -7\lambda^2 - 15\lambda - 8 \\ & 7\lambda^2 + 7\lambda \\ \hline & -8\lambda - 8 \end{array}$$

$\lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1)$

So $\lambda = -1$ has multiplicity 2 and $\lambda = 8$ does not.

Let $\lambda = -1$. Then

$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $2x_1 + x_2 + 2x_3 = 0$

$x_1 = -\frac{1}{2}x_2 - x_3$

So if $x_2 = 2$ and $x_3 = -2$, then $x_1 = 1$.

Then $\vec{x} = \begin{pmatrix} -\frac{1}{2}x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$

So $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are lin. indep. eigenvectors of $\lambda = -1$.

(14) $\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix}$

$= -(1-\lambda)(2-\lambda)(1+\lambda) + (1-\lambda) + 3(-1-\lambda) + 2 + 12 - 8(2-\lambda)$

$= (2-\lambda)(\lambda^2-1) + 1-\lambda - 3 - 3\lambda + 14 - 16 + 8\lambda$

$= 2\lambda^2 - \lambda^3 - 2 + \lambda - 4 + 4\lambda = -\lambda^3 + 2\lambda^2 + 5\lambda - 6$

$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

so $\lambda = 1$ is a solution.

Then
$$\begin{array}{r|l} & \lambda^2 - \lambda - 6 \\ \hline \lambda - 1 & \lambda^3 - 2\lambda^2 - 5\lambda + 6 \\ & -\lambda^3 + \lambda^2 \\ \hline & -\lambda^2 - 5\lambda + 6 \\ & \lambda^2 - \lambda \\ \hline & -6\lambda + 6 \\ & 0 \end{array}$$

$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$

So the eigenvalues are $\boxed{-2, 1, 3}$

If $\lambda = -2$ then

$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & -1 & 4 \\ 0 & 5 & -5 \\ 0 & 5/3 & -5/3 \end{pmatrix}$ so $x_2 = x_3$

and $3x_1 - x_2 + 4x_3 = 0$

so $x_1 = -x_3$. So $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda = -2$.

If one wants to make the top component 1, then the lin. indep. eigenvector are $\begin{pmatrix}1\\-2\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

(There are other ways of picking 2 lin. indep. vectors, for example $\begin{pmatrix}1\\-4\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ also work)

If $\lambda = 8$, then

$$\begin{pmatrix}-5 & 2 & 4\\ 2 & -8 & 2\\ 4 & 2 & -5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0 & -18 & 9\\ 2 & -8 & 2\\ 0 & 18 & -9\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{so} \quad \begin{aligned} 18x_2 - 9x_3 &= 0\\ 2x_1 - 8x_2 + 2x_3 &= 0\end{aligned}$$

$\boxed{x_2 = \frac{1}{2}x_3}$ $\qquad 2x_1 - 4x_3 + 2x_3 = 0$ so $x_1 = x_3$

So $\begin{pmatrix}2\\1\\2\end{pmatrix}$ is an eigenvector of $\lambda = 8$.

Then
$$\boxed{\vec{x} = c_1\begin{pmatrix}1\\-2\\0\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-t} + c_3\begin{pmatrix}2\\1\\2\end{pmatrix}e^{8t}}$$

when $\lambda = 1$ $\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 4 \\ 0 & -1/2 & 2 \\ 2 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 4 \\ 0 & 0 & 0 \\ 2 & 1 & -2 \end{pmatrix}$

so $-x_2 + 4x_3 = 0$ so $x_2 = 4x_3$

$2x_1 + x_2 - 2x_3 = 0$ so $2x_1 + 2x_3 = 0$ so $x_1 = -x_3$

Then $\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$ is an eigenvecto for $\lambda = 1$.

when $\lambda = 3$

$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & -5 \\ 2 & 1 & -4 \end{pmatrix}$

so $5x_1 - 5x_3 = 0$ so $x_1 = x_3$ so $x_1 = x_3$

$2x_1 + x_2 - 4x_3 = 2x_3 + x_2 - 4x_3 = x_2 - 2x_3 = 0$

so $x_2 = 2x_3$ so $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is the eigenvect

Then

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}}$$