7.6

(3) $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5$$

$$= -(2-\lambda)(2+\lambda) + 5$$

$$= -(4-\lambda^2) + 5$$

$$= \lambda^2 + 1$$

So $\lambda = i$ and $\lambda = -i$.

Let's focus on $\lambda = i$. Then $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $(2-i)x_1 - 5x_2 = 0$

So $x_1 = 1$ implies $x_2 = \frac{2-i}{5}$

Let $x_1 = 5$, then $x_2 = 2-i$. So the eigenvector is $\begin{pmatrix} 5 \\ 2-i \end{pmatrix}$.

Then $\begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{it} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 5\cos t + 5i\sin t \\ 2\cos t + \sin t - i\cos t + 2i\sin t \end{pmatrix}$

$$= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$$

Then $\boxed{\vec{x} = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}}$

(5) $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$ $\qquad \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda)+5$

$= -3 + 3\lambda - \lambda + \lambda^2 + 5 = \lambda^2 + 2\lambda + 2$

$\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \dfrac{-2 \pm \sqrt{4-8}}{2}$

$\lambda = -1 \pm \sqrt{-1} = -1 \pm i$

Take $\lambda = -1 + i$

Then $\begin{pmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{pmatrix} = \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}$

$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (2-i)x_1 - x_2 = 0$

so $x_2 = (2-i)x_1$

so $\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ is the eigenvector.

Then $\begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{-t}(\cos t + i \sin t)$

$= e^{-t}\begin{pmatrix} \cos t + i \sin t \\ 2\cos t + 2i \sin t - i\cos t + \sin t \end{pmatrix} = e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + e^{-t} i \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$

So $\boxed{\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}}$

(7) $x' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \vec{x}$ ... $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} + 0 = (1-\lambda)\left((1-\lambda)^2 - (1-\lambda)(-4)\right)$

$= (1-\lambda)\left((1-\lambda)^2+4\right)$

$= (1-\lambda)\left(\lambda^2-2\lambda+5\right) = 0$

so $\lambda = 1$ and $\lambda^2-2\lambda+5=0$ so $\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

From $\lambda = 1$ we get $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So $2x_1 - 2x_3 = 0 \Rightarrow x_1 = x_3$

$3x_1 + 2x_2 = 0 \Rightarrow x_2 = -\frac{3}{2}x_1 = -\frac{3}{2}x_3$

Let $x_3 = 2 \Rightarrow x_1 = +2$ and $x_2 = -3$. So $\begin{pmatrix} +2 \\ -3 \\ 2 \end{pmatrix}$ works.

From $\lambda = 1+2i$ we get $\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

so $x_1 = 0$ Then $2x_2 - 2ix_3 = 0$ so $x_2 = ix_3$, $x_3 = -ix_2$

Let $x_2 = 1 \Rightarrow x_3 = -i$

Then $e^{(1+2i)t}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} = e^t\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$

$= e^t\begin{pmatrix} 0 \\ \cos(2t) + i\sin(2t) \\ -i\cos(2t) + \sin(2t) \end{pmatrix} = e^t\begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} + ie^t\begin{pmatrix} 0 \\ \sin(2t) \\ -\cos(2t) \end{pmatrix}$

Therefore

$$\vec{x} = c_1\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}e^t + c_2e^t\begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} + c_3e^t\begin{pmatrix} 0 \\ \sin(2t) \\ -\cos(2t) \end{pmatrix}$$

7.8

(2) $x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x}$

$$\begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = (4-\lambda)(-4-\lambda) + 16$$

$$= -16 + \lambda^2 + 16 = \lambda^2$$

so $\lambda = 0$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4x_1 = 2x_2 \quad \text{so} \quad x_1 = \frac{x_2}{2}$$

So $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector.

So $c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a solution to the system.

Another solution should be of the form $\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \vec{v}$.

and $\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. So $4x_1 - 2x_2 = 1$
$8x_1 - 4x_2 = 2$

So $4x_1 - 2x_2 = 1$, then

$$x_1 = \frac{1}{4} + \frac{1}{2} x_2.$$

$$\text{So} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/4 + \frac{1}{2} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}.$$

Then the solution is of the form

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right)}$$

(3) $\begin{vmatrix} -3/2-\lambda & 1 \\ -1/4 & -1/2-\lambda \end{vmatrix} = (-3/2-\lambda)(-1/2-\lambda) + 1/4$

$= 3/4 + 2\lambda + \lambda^2 + 1/4 = \lambda^2+2\lambda+1$

$= (\lambda+1)^2.$

so $\boxed{\lambda = -1}$

$$\begin{pmatrix} -3/2+1 & 1 \\ -1/4 & -1/2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} -1/2 & 1 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-\frac{1}{2}x_1 + x_2 = 0$ so $x_1 = 2x_2$

eigenvector $= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Then one solution is $\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$.

Now for the other: $\begin{pmatrix} -1/2 & 1 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

so $-1/2\, x_1 + x_2 = 2$, so $-x_1 + 2x_2 = 4$

let $x_2 = k$. Then $x_1 = 2k - 4$

So $\begin{pmatrix} 2k-4 \\ k \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} k + \begin{pmatrix} -4 \\ 0 \end{pmatrix}.$

Then the general solution is

$$\boxed{\vec{x} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + c_2\begin{pmatrix} -4 \\ 0 \end{pmatrix} e^{-t}}$$

(5) $x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \vec{x}$.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & 1-\lambda \\ 0 & -1 \end{vmatrix}$$

$$= (1-\lambda)^3 - (1-\lambda) - 2(1-\lambda) - 2 = (1-\lambda)^3 - 3(1-\lambda) - 2.$$

$$= 1 - 3\lambda + 3\lambda^2 - \lambda^3 - 3 + 3\lambda - 2$$

$$= -\lambda^3 + 3\lambda^2 - 4.$$

$\lambda = -1$ is a root.

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 4 \\ \lambda+1 & \lambda^3 - 3\lambda^2 + 0\lambda + 4 \\ & -\lambda^3 - \lambda^2 \\ & -4\lambda^2 \\ & 4\lambda^2 + 4\lambda \\ & 4\lambda + 4 \end{array}$$

So the eigenvalues are $\boxed{\lambda = -1}$ and $\boxed{\lambda = 2}$

When $\lambda = -1$ we have $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $\boxed{x_2 = 2x_3}$

and hence $2x_1 + x_2 + x_3 = 0 \Rightarrow 2x_1 + 3x_3 = 0$

so $\boxed{x_1 = -\frac{3}{2}x_3}$

So the eigenvector is $\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$.

and the solution looks like $\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t}$

When $\lambda = 2$, we have

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $x_2 = -x_3$ and

$x_1 - x_2 - x_3 = 0$

so $x_1 = x_2 + x_3 = 0$

So the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. We don't have two lin. indep eigenvectors so we need to find a different solution.

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$x_1 = 1$ and $x_2 + x_3 = 1$ so $x_2 = 1 - x_3$

$$\begin{pmatrix} 1 \\ 1 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} x_3.$$

Then the general solution is

$$\vec{x} = c_1 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

5.1

(4) $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$

Ratio test: $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n}\right| = 2.$

So the radius of convergence is $\boxed{1/2}$.

So the interval of convergence is $(-1/2, 1/2)$.

(5) $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}(x+1/2)^n$

Then $x_0 = -1/2$ and $a_n = \frac{2^n}{n^2}$.

So the ratio test yields

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(n+1)^2}{2^n n^2}\right| = \lim_{n\to\infty}\left|2\frac{(n^2+2n+1)}{n^2}\right| = 2.$$

So the radius of convergence is $\boxed{1/2}$.

(That means the interval of convergence is $(-1, 0)$.)

(12) $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$, $f^{(n)}(x) = 0$ for $n \geq 3$.

$$x^2 = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + 0$$

$$= 1 + (-2)(x+1) + \frac{2(x+1)^2}{2!}$$

$$= \boxed{1 - 2(x+1) + (x+1)^2}$$

The radius of convergence is ∞.

(15) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ (famous Taylor series)

$= \sum_{n=0}^{\infty} x^n$

Ratio test $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{1}\right| = 1$.

So the radius of convergence is 1.

(18) $y = \sum_{n=0}^{\infty} a_n x^n$, so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$.

If $y'' = y$, then $(n+2)(n+1) a_{n+2} = a_n$

for $n \geq 0$. Therefore a_0 and a_1 are independent.

So $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$.

As a bonus, note that $a_2 = \frac{a_0}{2}$, $a_3 = \frac{a_1}{3 \cdot 2} = \frac{a_1}{3!}$,

$a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2} = \frac{a_0}{4!}$, $a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3!} = \frac{a_1}{5!}$,

so $a_{2n} = \frac{a_0}{(2n)!}$, while $a_{2n+1} = \frac{a_1}{(2n+1)!}$

so $y = \sum_{n=0}^{\infty} a_n x^n = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) +$

$a_1\left(x + \frac{x^3}{3!} + \ldots\right)$.

If $f(x) = 1 + \frac{x^2}{2!} + \cdots$ and $g(x) = x + \frac{x^3}{3!} + \cdots$

then $(f+g)(x) = e^x$ and $(f-g)(x) = e^{-x}$.

So $\boxed{y = a_0\left(\frac{e^x + e^{-x}}{2}\right) + a_1\left(\frac{e^x - e^{-x}}{2}\right)}$