$\boxed{5.2}$

(3) $y'' - xy' - y = 0, \quad x_0 = 1$

a) $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n$

$xy' = x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$

$= \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n + a_1$

So $y'' - xy' - y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$

$- a_1 - \sum_{n=0}^{\infty} a_n (x-1)^n$

$= 2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} \big((n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} - a_n \big)(x-1)^n$

$= (2a_2 - a_1 - a_0) + \sum_{n=1}^{\infty} (n+1)\big((n+2) a_{n+2} - a_{n+1} - a_n \big)(x-1)^n$

so $2a_2 = a_1 + a_0$ and

$(n+2) a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$.

The recurrence relation is $2a_2 = a_1 + a_0$ and

$a_{n+2} = \frac{a_n}{n+2} + \frac{a_{n+1}}{n+2}$

b) $a_2 = \frac{a_0}{2} + \frac{a_1}{2}$

$a_3 = \frac{a_1}{3} + \frac{a_2}{3} = \frac{a_1}{3} + \frac{a_0}{6} + \frac{a_1}{6} = \frac{a_0}{6} + \frac{a_1}{2}$

$a_4 = \frac{a_3}{4} + \frac{a_2}{4} = \frac{a_0}{24} + \frac{a_1}{8} + \frac{a_0}{8} + \frac{a_1}{8} = \frac{a_0}{6} + \frac{a_1}{4}$

$a_5 = \frac{a_3}{5} + \frac{a_4}{5} = \frac{a_0}{30} + \frac{a_1}{20} + \frac{a_0}{30} + \frac{a_1}{10} = \frac{a_0}{15} + \frac{3a_1}{20}$

$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \dots$

$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \frac{a_1}{2}(x-1)^2 + \dots$

$= a_0\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \dots\right)$

$+ a_1\left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \dots\right)$

c) $y_1(1) = 1, \quad y_1'(1) = 0$

$y_2(1) = 0, \quad y_2'(1) = 1 \quad$ so $W(y_1, y_2)(1) = 1$.

so y_1, y_2 are fundamental solutions.

d) Doesn't seem to have an easy translation for y_1, y_2.

$y(x) = a_0 y_1(x) + a_1 y_2(x)$

(5) $(1-x)y'' + y = 0$, $x_0 = 0$

$P(x) = 1-x$, so the answer will exist in the interval $(-\infty, 1)$, since $P(x) \neq 0$ (and $x_0 = 0$, so the interval should include 0).

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(1-x)y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^n$$

$$= \sum_{n=0}^{\infty} (n+1)\left((n+2) a_{n+2} - n a_{n+1}\right) x^n$$

Since $(1-x)y'' = -y$, then

$$(n+1)\left((n+2) a_{n+2} - n a_{n+1}\right) = -a_n$$

So $$\boxed{(n+2) a_{n+2} - n a_{n+1} + \frac{a_n}{n+1} = 0}$$

b) $(n+2)a_{n+2} = n a_{n+1} - \frac{a_n}{n+1}$

$$a_{n+2} = \frac{n}{n+2} a_{n+1} - \frac{1}{(n+1)(n+2)} a_n$$

$$a_2 = -\frac{1}{2} a_0, \quad a_3 = \frac{1}{3} a_2 - \frac{1}{3\cdot 2} a_1 = -\frac{1}{6} a_0 - \frac{1}{6} a_1$$

$$a_4 = \frac{1}{2} a_3 - \frac{1}{4\cdot 3} a_2 = \frac{1}{2}\left(-\frac{1}{6} a_0 - \frac{1}{6} a_1\right) - \frac{1}{4\cdot 3}\left(\frac{-a_0}{2}\right)$$

$$= \frac{a_0}{4!} - \frac{1}{12} a_0 - \frac{1}{12} a_1 = -\frac{a_0}{24} - \frac{1}{12} a_1$$

$$a_5 = \frac{3}{5} a_4 - \frac{1}{20} a_3 = -\frac{3}{120} a_0 - \frac{3}{60} a_1 + \frac{1}{120} a_0 + \frac{1}{120} a_1 = -\frac{1}{60} a_0 - \frac{1}{24} a_1$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_0 x^3 - \frac{1}{6} a_1 x^3 - \frac{1}{24} a_0 x^4 - \frac{1}{12} a_1 x^4 + \ldots$$

$$= a_0\left(1 - \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{1}{24} x^4 + \ldots\right)$$

$$+ a_1\left(x - \frac{1}{6} x^3 - \frac{1}{12} x^4 - \frac{1}{24} x^5 + \ldots\right)$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

c) $y_1(0) = 1, \quad y_1'(0) = 0$

$y_2(0) = 0, \quad y_2'(0) = 1$ so $W(y_1, y_2)(0) = 1$

so y_1 and y_2 are fundamental solutions.

d) No easy formula for $y_1(x)$, and $y_2(x)$.

15a) $y'' - xy' - y = 0 \qquad y(0) = 2, \; y'(0) = 1$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy' = x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n \; \left(\text{because when } n=0,\; n a_n = 0\right)$$

So $y'' - xy' - y = 0$ implies

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} - (n+1) a_n\right) x^n$$

so $(n+2)(n+1) a_{n+2} = (n+1) a_n$

so $(n+2) a_{n+2} = a_n$

so $\boxed{a_{n+2} = \frac{1}{n+2} a_n}$

Then $a_0 = a_0,\; a_1 = a_1,\; a_2 = \frac{1}{2} a_0,\; a_3 = \frac{1}{3} a_1,\; a_4 = \frac{1}{2 \cdot 4} a_0,\; a_5 = \frac{1}{3 \cdot 5} a_1, \ldots$

So $y(x) = a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \frac{x^6}{2 \cdot 4 \cdot 6} + \ldots\right)$

$\qquad + a_1 \left(x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 3} + \ldots\right)$

$y(0) = 2$, so $a_0 = 2$. $y'(0) = 1$ so $a_1 = 1$.

So $\boxed{y(x) = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \ldots}$

(16a) $(2+x^2)y'' - xy' + 4y = 0$, $\quad y(0) = -1$, $y'(0) = 3$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{so} \quad xy' = \sum_{n=0}^{\infty} n a_n x^n.$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad \text{so}$$

since $n(n-1) = 0$ for $n=0$ and $n=1$

$$(2+x^2)y'' = 2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$= \sum_{n=0}^{\infty} \left(2(n+2)(n+1)a_{n+2} + n(n-1)a_n\right) x^n.$$

$$(2+x^2)y'' - xy' + 4y = \sum_{n=0}^{\infty} \left(2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n\right) x^n$$

So $\quad 2(n+2)(n+1)a_{n+2} = (n - 4 - n(n-1))a_n = -(n^2-2n+4)a_n$

$$\boxed{a_{n+2} = \frac{-(n^2-2n+4)}{2(n+2)(n+1)} a_n}$$

$y(0) = a_0$, $\quad y'(0) = a_1 \quad$ so $\quad a_0 = -1$, $y'(0) = 3$

$a_0 = -1$, $\quad a_2 = \dfrac{-(+4)}{4} a_0 = 1$, $\quad a_4 = \dfrac{-(2^2-4+4)}{2(4)(3)} a_2 = \dfrac{-1}{6}$

$a_1 = 3$, $\quad a_3 = \dfrac{-(1-2+4)}{2\cdot 3\cdot 2} a_1 = \dfrac{-3}{12} a_1 = \dfrac{-3}{4}$

$$\boxed{y(x) = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \cdots .}$$

$\boxed{5.3}$

(2) $y'' + (\sin x)y' + (\cos x)y = 0, \quad y(0)=0, \ y'(0)=1.$

Suppose $y = \phi(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution to the DFQ.

Then $\phi''(0) = 2a_2$, $\phi'''(0) = 3!\,a_3$, $\phi^{(4)}(0) = 4!\,a_4$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = 2a_2 + 3\,2a_3 x + 4\cdot 3a_4 x^2 + \dots$$

$$(\sin x)y' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(a_1 + 2a_2 x + 3a_3 x^2 + \dots\right)$$

$$= a_1 x + 2a_2 x^2 + \left(3a_3 - \frac{1}{3!}\right)x^3 + \left(4a_4 - \frac{2}{3!}a_2\right)x^4 + \dots$$

$$(\cos x)y = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots\right)$$

$$= a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 - \frac{a_1}{2}\right)x^3 + \left(a_4 - \frac{a_2}{2} + \frac{a_0}{4!}\right)x^4 + \dots$$

So $y'' + (\sin x)y' + (\cos x)y = 0$ implies

$$\left(2a_2 + 6a_3 x + 12a_4 x^2 + \dots\right) + \left(a_1 x + 2a_2 x^2 + \left(3a_3 - \frac{1}{3!}\right)x^3 + \dots\right) +$$
$$\left(a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 - \frac{a_1}{2}\right)x^3 + \dots\right)$$

$$= \left(2a_2 + a_0\right) + \left(6a_3 + 2a_1\right)x + \left(12a_4 + 2a_2 + a_2 - \frac{a_0}{2}\right)x^2 + \dots$$

Since $y(0)=0$, then $a_0 = 0$. Since $y'(0)=1$, then $a_1 = 1$.

So $2a_2 + a_0 = 0$ implies $a_2 = 0$.
$6a_3 + 2a_1 = 0$ implies $a_3 = -\frac{2}{6} = -1/3$

$12a_4 + 2a_2 + a_2 - \frac{a_0}{2} = 0$ implies $a_4 = 0$.

Therefore $\boxed{\phi''(0) = 0, \quad \phi'''(0) = 3!\left(-\frac{1}{3}\right) = -2, \quad \phi^{(4)}(0) = 0.}$

(7) $(1+x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \ x_0 = 2.$

Let's do it for $x_0 = 0$ first.

$$p(x) = \frac{4x}{1+x^3}, \quad q(x) = \frac{1}{1+x^3}$$

The lower bound for the radius of convergence is at least the minimum of the radii of convergence of p and q.

$1+x^3=0$ implies $x^3=-1$, so $x=-1$ is one root.

$(1+x)(x^2-x+1)$. The other roots are $x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$.

The distance from $x_0 = 0$ to $x=-1$ is 1.

From 0 to $\frac{1 \pm \sqrt{3}i}{2}$ is $\left| \frac{1 \pm \sqrt{3}i}{2} \right| = \frac{\sqrt{(1 \pm \sqrt{3}i)(1 \mp \sqrt{3}i)}}{2}$

$= \frac{\sqrt{4}}{4} = 1$. So the radius of convergence is at least $\boxed{1}$ for $x_0 = 0$.

For $x_0 = 2$ we consider the distances to -1, $\frac{1+\sqrt{3}i}{2}$ and $\frac{1-\sqrt{3}i}{2}$.

To -1, the distance is 3.

To $\frac{1+\sqrt{3}i}{2}$ the distance is $\left| -1 - \frac{1+\sqrt{3}i}{2} \right| = \left| \frac{-3-\sqrt{3}i}{2} \right|$

$= \frac{\sqrt{(-3-\sqrt{3}i)(-3+\sqrt{3}i)}}{2} = \frac{\sqrt{12}}{2} = \sqrt{3}$

To $\frac{1-\sqrt{3}i}{2}$ the distance is $\left| -1 - \frac{1-\sqrt{3}i}{2} \right| = \left| \frac{-3+\sqrt{3}i}{2} \right|$

$= \frac{\sqrt{(-3-\sqrt{3}i)(-3+\sqrt{3}i)}}{2} = \sqrt{3}$.

So the minimum radius of convergence is $\boxed{\sqrt{3}}$.

$\boxed{5.4}$

(2) $(x+1)^2 y'' + 3(x+1)y' + \frac{3}{4}y = 0$

Let $y = (x+1)^r$, then $y' = r(x+1)^{r-1}$, $y'' = r(r-1)(x+1)^{r-2}$

So $(x+1)^2 y'' + 3(x+1)y' + \frac{3}{4}y = r(r-1)(x+1)^r + 3r(x+1)^r + \frac{3}{4}(x+1)^r$

So $r(r-1) + 3r + \frac{3}{4} = 0$.

$r^2 + 2r + 3/4 = 0$

$(r+\frac{3}{2})(r+\frac{1}{2}) = 0$ Then $r = -3/2$, $r = -1/2$.

So $\boxed{y(x) = c_1(x+1)^{-\frac{3}{2}} + c_2(x+1)^{-1/2}}$ is the general solution.

(3) $x^2y'' - 3xy' + 4y = 0$

Let $y = x^r$, so $r(r-1) - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

so $(r-2)^2 = 0$. So $\boxed{r = 2}$

Then $\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln(x).}$

(11) $x^2y'' + 2xy' + 4y = 0$

So $r(r-1) + 2r + 4 = 0$, so $r^2 + r + 4 = 0$

$r = \frac{-1 \pm \sqrt{1-16}}{2} = \frac{-1 \pm \sqrt{15}\, i}{2}$

Consider $\frac{-1 + \sqrt{15}\, i}{2}$.

$x^{\frac{-1+\sqrt{15}i}{2}} = e^{-\frac{1}{2}\ln x} e^{i\frac{\sqrt{15}}{2}\ln x} = x^{-1/2}\left(\cos\left(\frac{\sqrt{15}}{2}\ln x\right) + i\sin\left(\frac{\sqrt{15}}{2}\ln x\right)\right)$

So $x^r = x^{-1/2}\cos\left(\frac{\sqrt{15}}{2}\ln(x)\right) + i\, x^{-1/2}\sin\left(\frac{\sqrt{15}}{2}\ln(x)\right)$

Therefore the general solution is of the form:

$$\boxed{y(x) = c_1 x^{-1/2}\cos\left(\frac{\sqrt{15}}{2}\ln(x)\right) + c_2 x^{-1/2}\sin\left(\frac{\sqrt{15}}{2}\ln(x)\right)}$$

(16) $x^2y'' + 3xy' + 5y = 0$, $y(1) = 1$, $y'(1) = -1$.

$r(r-1) + 3r + 5 = 0$, so $r^2 + 2r + 5 = 0$, so

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$x^{-1+2i} = x^{-1}\cdot x^{2i} = x^{-1}\cdot e^{2i\ln x} = x^{-1}\left(\cos(2\ln(x)) + i\sin(2\ln(x))\right)$$

So $y(x) = c_1 x^{-1}\left(\cos(2\ln(x))\right) + c_2 x^{-1}\sin(2\ln(x))$

(Recall $\ln(1) = 0$) $y(1) = c_1 = 1$, so $c_1 = 1$.

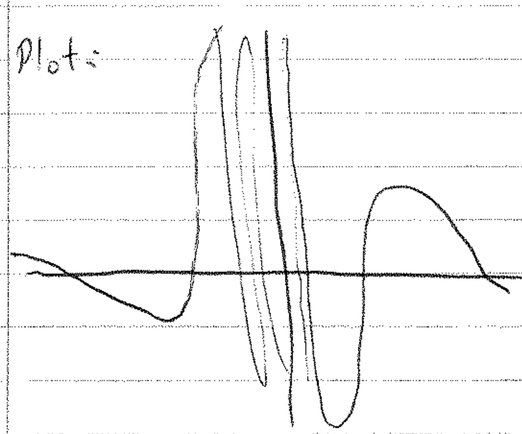
$$y'(x) = -c_1 x^{-2}\left(\cos(2\ln(x))\right) - \frac{2}{x}c_1 x^{-1}\sin(2\ln(x))$$
$$- c_2 x^{-2}\sin(2\ln x) + \frac{2}{x}c_2 x^{-1}\cos(2\ln(x))$$

$y'(1) = -c_1 + 2c_2 = -1$

so $2c_2 = 0$, so $c_2 = 0$.

So $\boxed{y(x) = \frac{\cos(2\ln(x))}{x}}$

Plot:



Lots of oscillation near $x=0$ with the amplitude going to infinity.