# Practice Exam Solutions

(1) a) Let $Q(t)$ the amount of grams of the chemical after $t$ minutes.

rate in times concentration

rate out times the concentration

$$\frac{dQ}{dt} = 2\left(\frac{1}{4}\left(1+\frac{t}{2}\right)\right) - \frac{2Q}{100}$$

$$= \frac{1}{2} + \frac{t}{4} - \frac{Q}{50}$$

$$Q' + \frac{1}{50}Q = \frac{1}{2} + \frac{t}{4}.$$

b)

$$\mu Q' + \frac{1}{50}\mu Q = \mu\left(\frac{1}{2}+\frac{t}{4}\right).$$

$$\frac{d(\mu Q)}{dt} = \mu Q' + \mu' Q$$ so we want

$\mu' Q = \frac{1}{50}\mu Q$ so $\frac{\mu'}{\mu} = \frac{1}{50}$ so $\mu = e^{\frac{1}{50}t}$.

Then $$e^{\frac{1}{50}t} Q = \int \left(\frac{1}{2}+\frac{t}{4}\right) e^{\frac{1}{50}t}\,dt$$

$$= \frac{1}{2}\int e^{\frac{1}{50}t} + \frac{1}{4}\int t e^{\frac{1}{50}t}\,dt + C$$ $u = t$, $dv = e^{\frac{1}{50}t}$, $du = dt$, $v = 50e^{\frac{1}{50}t}$

$$= 25e^{\frac{1}{50}t} + \frac{1}{4}\left(50te^{\frac{1}{50}t} - \int 50e^{\frac{1}{50}t}dt\right) + C$$

$$= 25e^{\frac{1}{50}t} + \frac{25}{2}te^{\frac{1}{50}t} - \frac{50\cdot 50}{4}e^{\frac{1}{50}t} + C$$

So $$Q(t) = 25 + \frac{25}{2}t - 625 + Ce^{-\frac{1}{50}t} = \frac{25}{4}t - 600 + Ce^{-\frac{1}{50}t}.$$

$Q(0) = 50$ so $C - 600 = 50$ so $C = 650$

$$\boxed{Q(t) = \frac{25}{2}t - 600 + 650e^{-\frac{t}{50}}}$$

(2) $y' + 2y = te^{-2t}$, $y(1) = 0$.

$\mu y' + 2\mu y = \mu t e^{-2t}$

Then $\mu' = 2\mu$ so $\mu = e^{2t}$.

Then $e^{2t} y = \int e^{2t} t e^{-2t} dt = \int t\, dt = \frac{t^2}{2} + C$

So $y(t) = \frac{t^2}{2} e^{-2t} + Ce^{-2t}$.

$y(1) = \frac{1}{2} e^{-2} + Ce^{-2} = 0$ so $C = -\frac{1}{2}$.

Then $y(t) = \frac{t^2}{2} e^{-2t} - \frac{1}{2} e^{-2t} = \left(\frac{t^2 - 1}{2}\right) e^{-2t}$.

(3) $\frac{dy}{dx} = \frac{x^2}{y}$. $dy\, y = x^2 dx$

$\frac{y^2}{2} = \frac{x^3}{3} + C$.

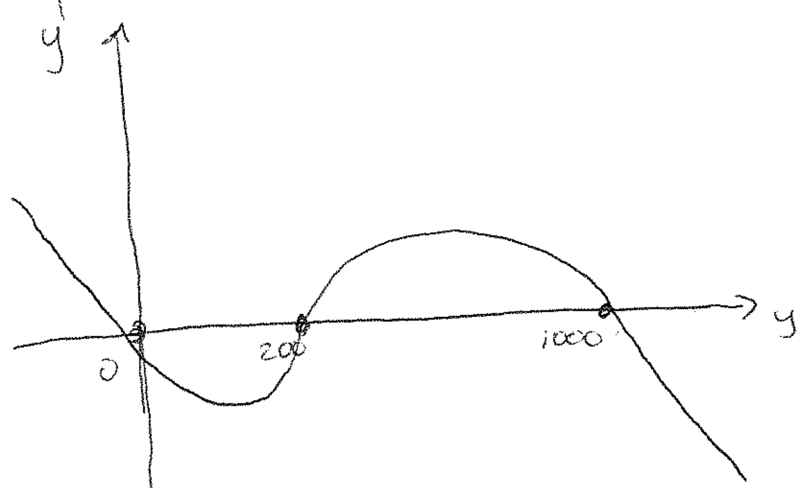
So $y^2 = \frac{2}{3} x^3 + C$

$$\boxed{y = \pm\sqrt{\frac{2}{3} x^3 + C}}$$

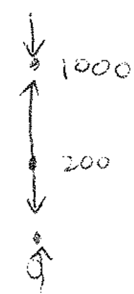
(4) $y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$. $\frac{2t}{4-t^2}$ and $\frac{3t^2}{4-t^2}$ are continuous as long as $4 - t^2 \neq 0$, i.e. $t \neq \pm 2$. Then the possible intervals are $(-\infty, -2), (-2, 2), (2, \infty)$. But $y(1) = -3$ implies 1 must be in the interval. So it must be $\boxed{(-2, 2)}$

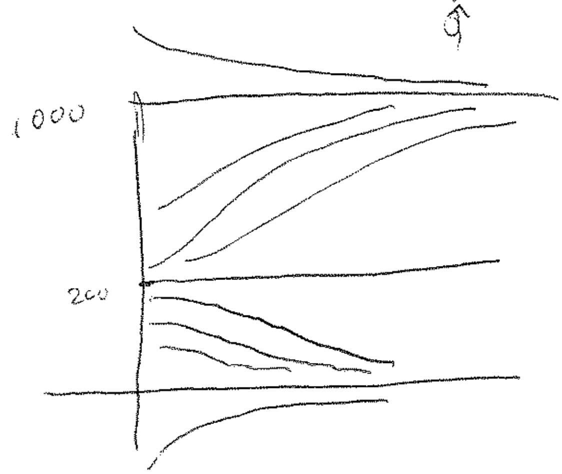
(5) $y' = -y\left(1-\frac{y}{a}\right)\left(1-\frac{y}{1000}\right)$.

a) $a = 200$.



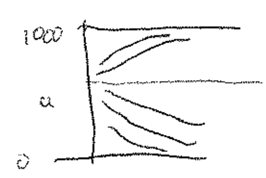

Phase line:



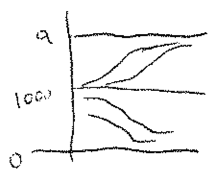

Integral curves

b) If ~~α~~ $a < 1000$ then



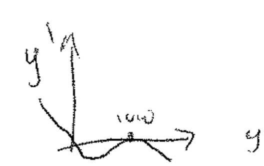

1000 and 0 are stable solutions and $a$ is unstable

If $a > 1000$ then



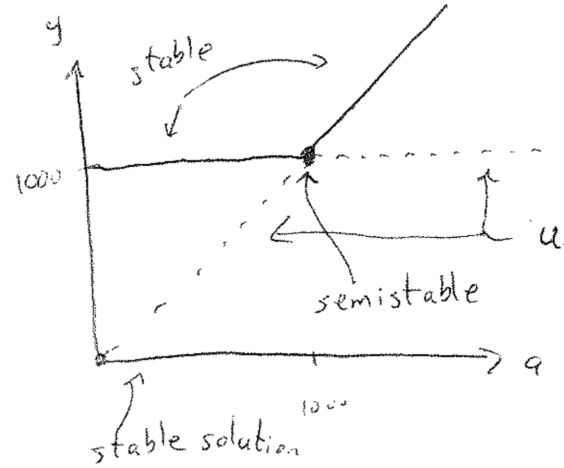

$a$ and 0 are stable while 1000 is unstable.

If $a = 1000$ then




so 0 is stable while 1000 is semistable.

c)




(Plot of the equilibrium solutions as a changes)

⑥ a) $y'' - 6y' + 18y = 0$

$r^2 - 6r + 18 = 0$

$$r = \frac{6 \pm \sqrt{36 - 4 \cdot 18}}{2} = \frac{6 \pm \sqrt{-36}}{2} = \frac{6 \pm 6i}{2} = 3 \pm 3i$$

so $\boxed{y = c_1 e^{3t} \cos 3t + c_2 e^{3t} \sin(3t)}$

b) $4y'' - 4y' + 3y = 0$

$4r^2 - 4r + 3 = 0$

$$r = \frac{4 \pm \sqrt{16 - 4(4)(3)}}{8} = \frac{4 \pm \sqrt{-32}}{8} = \frac{1}{2} \pm i \frac{\sqrt{2}}{2}$$

so

$$\boxed{y(t) = c_1 e^{t/2} \cos\left(\frac{\sqrt{2}}{2} t\right) + c_2 e^{t/2} \sin\left(\frac{\sqrt{2}}{2} t\right)}$$

c) $y'' - 6y' + 18y = 3e^{3t}$.

$r^2 - 6r + 18 = 0$ so $y = c_1 e^{3t} \cos(3t) + c_2 e^{3t} \sin(3t)$ is a solution to the homogeneous equation.

Let $y = Ae^{3t}$. Then $y' = 3Ae^{3t}$ and $y'' = 9Ae^{3t}$, so

$$9Ae^{3t} - 18Ae^{3t} + 18Ae^{3t} = 9Ae^{3t} = 3e^{3t} \quad \text{so}$$

$A = 1/3$

Then $y = \frac{1}{3}e^{3t}$ is a particular solution.

Then $$\boxed{y(t) = c_1 e^{3t}\cos(3t) + c_2 e^{3t}\sin(3t) + \frac{1}{3}e^{3t}}$$

d) $y'' + y = \tan t$

$y'' + y = 0$ implies $r^2 + 1 = 0$ so $r = \pm i$ so

$y(t) = c_1 \cos t + c_2 \sin(t)$.

Now let $y(t) = u_1(t)\cos t + u_2(t)\sin t$.

Then $y'(t) = u_1'(t)\cos t - u_1(t)\sin t + u_2'(t)\sin t + u_2(t)\cos t$

Assume $u_1'(t)\cos t + u_2'(t)\sin t = 0$.

Then $y'(t) = u_2(t)\cos t - u_1(t)\sin t$.

So $y''(t) = u_2'(t)\cos t - u_2(t)\sin t - u_1'(t)\sin t - u_1(t)\cos t$.

So $y'' + y' = u_2'(t)\cos t - \cancel{u_2(t)\sin t} - u_1'(t)\sin t - \cancel{u_1(t)\cos t} + \cancel{u_1(t)\cos t} + \cancel{u_2(t)\sin t}$

$= -u_1'(t)\sin t + u_2'(t)\cos t = \tan t$.

So
$$-u_1'(t)\sin t + u_2'(t)\cos t = \tan t$$
$$u_1'(t)\cos t + u_2'(t)\sin t = 0$$

So
$$-u_1'(t)\sin t\cos t + u_2'(t)\cos^2 t = \sin t$$
$$u_1'(t)\sin t\cos t + u_2'(t)\sin^2 t = 0$$

So $u_2'(t)(\sin^2 t + \cos^2 t) = \sin t$, so $u_2'(t) = \sin t$.

So $\boxed{u_2(t) = -\cos t}$

Now $u_1'(t)\cos t = -u_2'(t)\sin t = -\sin^2 t = \cos^2 t - 1$

So $u_1'(t) = \cos t - \sec t$.

Then $u_1(t) = \sin t - \int \sec t = \sin t - \ln\left(\frac{\sin\frac{t}{2} + \cos\frac{t}{2}}{\cos\frac{t}{2} - \sin\frac{t}{2}}\right)$

So

$$y(t) = c_1\cos t + c_2\sin t + \sin t\cos t - \cos t\ln\left(\frac{\sin\frac{t}{2}+\cos\frac{t}{2}}{\cos\frac{t}{2}-\sin\frac{t}{2}}\right) - \sin t\cos t$$

$$y(t) = c_1\cos t + c_2\sin t - \cos t\ln\left(\frac{\sin(\frac{t}{2})+\cos(\frac{t}{2})}{\cos(\frac{t}{2})-\sin(\frac{t}{2})}\right).$$

7 a) $y = t^n$, $y' = nt^{n-1}$, $y'' = n(n-1)t^{n-2}$

$$t^2y'' + 3ty' - 3y = n(n-1)t^n + 3nt^n - 3t^n$$
$$= (n^2 - n + 3n - 3)t^n = (n^2+2n-3)t^n = 0$$

So $n^2+2n-3=0$

so $(n+3)(n-1)=0$, so $n=-3$ and $n=1$.

b) $y(t) = c_1 t + c_2 t^{-3}$.

Since $W(t, t^{-3})(t) = t(t^{-3})' - t'(t^{-3})$
$$= -3tt^{-4} - t^{-3}$$
$$= -4t^{-3} \neq 0 \text{ unless } t=0.$$

8 $y'' - 2y' + y = 3 + e^{2t}$.

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$, so $r=1$.

Then $y(t) = c_1e^t + c_2te^t$ solves the homogeneous equation.

Let $y = Ate^{2t} + Be^{2t}$.

$y' = Ae^{2t} + 2Ate^{2t} + 2Be^{2t}$

$y'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} + 4Be^{2t}$

$y'' = 4Ae^{2t} + 4Be^{2t} + 4Ate^{2t}$

Then $y'' - 2y' + y = e^{2t}(4A + 4B - 2(\cancel{A} + 2B) + B) + te^{2t}(4A - 4A + A)$

$= Ate^{2t} + (2A + B)e^{2t}$.

So $A = 3$ and $2A + B = 0$ so $B = -6$.

Then $y(t) = c_1 e^t + c_2 te^t + 3te^{2t} - 6e^{2t}$.

$y'(t) = c_1 e^t + c_2 e^t + c_2 te^t + 3e^{2t} + 6te^{2t} - 12e^{2t}$

Since $y(0) = 2$ then $2 = c_1 - 6$ so $\boxed{c_1 = 8}$

Since $y'(0) = 4$ then $4 = c_1 + c_2 + 3 - 12 = 8 + c_2 + 3 - 12 = c_2 - 1$

so $\boxed{c_2 = 5}$

Then

$$\boxed{y(t) = 8e^t + 5te^t + 3te^{2t} - 6e^{2t}}$$