Practice Exam 1

1. Let $f : A \to B$ and $g : B \to C$ be maps.

   (a) If f and g are both one-to-one functions, show that $g \circ f$ is one-to-one.

   (b) If $g \circ f$ is onto, show that g is onto.

   (c) If $g \circ f$ is one-to-one, show that f is one-to-one.

   (d) If $g \circ f$ is one-to-one and f is onto, show that g is one-to-one.

   (e) If $g \circ f$ is onto and g is one-to-one, show that f is onto.

2. Determine whether or not the following relations are equivalence relations on the given set. If the relation is an equivalence relation, describe the partition given by it. If the relation is not an equivalence relation, state why it fails to be one.

   (a) $x \sim y$ in $\mathbb{R}$ if $x \geq y$

   (b) $m \sim n$ in $\mathbb{Z}$ if $mn > 0$

   (c) $x \sim y$ in $\mathbb{R}$ if $|x - y| \leq 4$

   (d) $m \sim n$ in $\mathbb{Z}$ if $m \equiv n \pmod 6$

3. For each of the following pairs of numbers a and b, calculate $\gcd(a, b)$ and find integers r and s such that $\gcd(a, b) = ra + sb$.

   (a) 14 and 39

   (b) 234 and 165

4. Which of the following associative multiplication tables defined on the set $G = \{a, b, c, d\}$ form a group? Support your answer in each case.

   (a)

$\circ$	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

   (b)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

   (c)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

(d)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	b	a	d
d	d	d	b	c

5. Let $S = \mathbb{R} \setminus \{-1\}$ and define a binary operation on S by $a * b = a + b + ab$. Prove that $(S, *)$ is an abelian group.

6. Find all the subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Use this information to show that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not the same group as $\mathbb{Z}_9$.

7. Let $n = 0, 1, 2, \ldots$ and $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$. Prove that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. Show that these subgroups are the only subgroups of $\mathbb{Z}$.

8. Prove or disprove each of the following statements.

   (a) $\mathbb{Z}_8^\times$ is cyclic.

   (b) All of the generators of $\mathbb{Z}_{60}$ are prime.

   (c) $\mathbb{Q}$ is cyclic.

   (d) If every proper subgroup of a group G is cyclic, then G is a cyclic group.

   (e) A group with a finite number of subgroups is finite.

9. Find the order of each of the following elements.

   (a) $5 \in \mathbb{Z}_{12}$

   (b) $\sqrt{3} \in \mathbb{R}$

   (c) $\sqrt{3} \in \mathbb{R}^*$

   (d) $-i \in \mathbb{C}^*$

10. Prove that $\mathbb{Z}_p$ has no nontrivial subgroups if p is prime.