A birthday in St. Petersburg

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Consider the following question: How many flips of the coin are required to get the same number of Heads as Tails? Sometimes, you only need two flips (when you get Heads-Tails or Tails-Heads), sometimes you need 4 ($HHTT$ or $TTHH$), or 6 ($HHTHTT, HHHTTT, TTTHHH$, or $TTHTHH$), et cetera. We will use this problem as inspiration to revisit two famous paradoxes, the Birthday Paradox and the St. Petersburg Paradox.

St. Petersburg Paradox Revisited

The St. Petersburg paradox, as written in [**3**], is as follows: "A fair coin is flipped and the player is paid \$2 if a head occurs on the first toss, \$4 if a head first appears on the second toss and, in general, $\$2^k$ if Heads first appears on the k-th toss. What would you be willing to pay for the privilege of playing this game?". The reason we consider it a paradox is that the expected value of the payment is infinite, indeed the probability that the first H occurs in the k-th toss is 2^{-k} because the only way it occurs is if we have the sequence $TTT\cdots TH$ with $k-1$ Tails at the beginning. This implies that the expected payment is

$$2\left(\frac{1}{2}\right)+2^2\left(\frac{1}{2}\right)^2+\ldots=1+1+\ldots=\infty.$$

The paradox was introduced by Daniel Bernoulli in [**2**] and has been heavily studied over the years. Some references include [**7**], [**10**], [**1**], and [**6**].

The game we will play is the following:

A fair coin is flipped and the player is paid $\$2$ *if after two flips the coin landed Heads once and Tails once,* $\$4$ *if it takes four coin flips to get the same number of Heads as Tails, and in general,* $\$2k$ *if it takes* $2k$ *flips for the coin to land the same number of times Heads as Tails. How much would a person be willing to pay to play this game?*

Let X be the random variable representing the payment. As in the St. Petersburg paradox, we find that the expected payoff diverges to infinity. In this short article, we'll prove that $\mathbb{E}[X]=\infty$ by first calculating the probability that $X=2k$ for a positive integer k (note that X is always even).

Before we get to our main results, let's remember a classical combinatorics result (which can be found in [**11**]).

Lemma 1. *The number of paths from $(0,0)$ to (n,n) using only unit steps to the right or up that don't go above the diagonal from $(0,0)$ to (n,n) is the n-th Catalan number:*

$$\frac{1}{n+1}\binom{2n}{n}.$$

We can use this lemma to prove the following theorem.

Theorem 1. *Let k be a positive integer. Then*

$$\mathbb{P}[X = 2k] = \frac{2}{4^k k}\binom{2k-2}{k-1}.$$

Proof. Note that X represents the number of coin flips required to get the same number of Heads as Tails. Therefore, we want to figure out in how many ways we can get k Heads and k Tails without having the number of Heads equal the number of Tails happen beforehand. We can think of this as the number of paths from $(0,0)$ to (k,k) using only steps to the right ("Heads") and steps upwards ("Tails") which don't intersect the diagonal from $(0,0)$ to (k,k). This is almost the situation of Lemma 1, however in this case we can't touch the diagonal, while the Lemma allows "touching" the diagonal as long as it's not crossed.

If the first coin flip is Heads, then we're at $(1,0)$. To avoid touching the diagonal from $(0,0)$ to (k,k), a path must move from $(1,0)$ to $(k,k-1)$ but not cross the diagonal from $(1,0)$ to $(k,k-1)$ (see Figure 1). But considering the subsquare of side-length $k-1$ with vertices $(1,0),(1,k-1),(k,0),(k,k-1)$ reveals that, by Lemma 1, the number of such paths is the $(k-1)$th Catalan number $\frac{1}{k}\binom{2k-2}{k-1}$. Given that the first flip could be Tails, we multiply this number of paths by 2. Since there are $2^{2k} = 4^k$ possible outcomes of flipping a coin $2k$ times, the formula for $\mathbb{P}[X = 2k]$ follows. ■

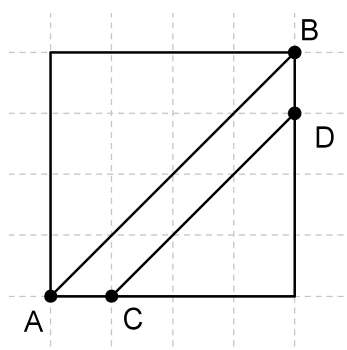

Figure 1. Example of $k = 4$. The diagonal $\overline{AB}$ is to be avoided, hence the diagonal $\overline{CD}$ is not to be crossed.

Remark. This result is known as the "first return on a symmetric one dimensional random walk". Most proofs we found of this result (see [**4**], [**5**]) use generating functions. One proof that uses Catalan numbers can be found in [**8**].

Now note that we can express $\mathbb{P}[X = 2k]$ as

$$\frac{2}{4^k k}\binom{2k-2}{k-1} = \frac{1}{4^{k-1}}\binom{2k-2}{k-1} - \frac{1}{4^k}\binom{2k}{k},$$

and so

$$\sum_{k=1}^{\infty} \frac{2}{4^k k}\binom{2k-2}{k-1} = \sum_{k=1}^{\infty}\left(\frac{1}{4^{k-1}}\binom{2k-2}{k-1} - \frac{1}{4^k}\binom{2k}{k}\right).$$

This is a telescoping sum, and the subtracted term goes to 0 by Stirling's formula, which states that $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, and thus we have

$$\sum_{k=1}^{\infty} \frac{2}{4^k k}\binom{2k-2}{k-1} = 1.$$

This implies the following:

Corollary 1. *The probability that eventually the number of Heads will equal the number of Tails is 1. In other words, the probability that the process terminates is 1, which implies* $\mathbb{P}[X = \infty] = 0$.

Remark. This corollary was first proven by Pólya in [**9**] in a more general setting.

We are now ready to prove that our St. Petersburg variant game has infinite expected payout.

Theorem 2. *Let X be the number of coin flips it takes to get the same number of Heads as Tails. Then the expected value of X is*

$$\mathbb{E}[X] = \infty.$$

Proof. Since $\mathbb{P}[X = \infty] = 0$, the expected value of X is

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty}(2k)\mathbb{P}[X = 2k] = \sum_{k=1}^{\infty}(2k)\frac{2}{4^k k}\binom{2k-2}{k-1}\\ &= \sum_{k=1}^{\infty}\frac{1}{4^{k-1}}\binom{2k-2}{k-1} = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}.\end{aligned}$$

Since the sum of the $2n$-th row of Pascal's triangle is 4^n, and $\binom{2n}{n}$ is the biggest of the $2n+1$ terms in the n-th row of Pascal's triangle, then $\binom{2n}{n} \geq \frac{1}{2n+1}4^n$. Therefore

$$\mathbb{E}[X] \geq \sum_{n=0}^{\infty}\frac{1}{2n+1} > \sum_{n=1}^{\infty}\frac{1}{2n} = \infty.$$

■

Birthday Paradox Revisited

Now suppose you have a class of n students and you ask each of them to flip a coin until they each have the same number of Heads as Tails. Consider the maximum among all of the students. As shown above, the expected number of coin flips for each student is infinite. However, in practice, the process will end. Anecdotal evidence suggests that the maximum number of coin flips (among the results for the students) is surprisingly large (generally in the hundreds) even in small classes (a dozen students). In this section we'll analyze this question.

We'll start by calculating the probability that at least one of the students flips a coin m or more times (we may assume m is even, since the number of coin flips is never odd when the game terminates). This probability is

$$1 - \left(\sum_{k=1}^{\frac{m}{2}-1} \frac{2}{4^k k} \binom{2k-2}{k-1} \right)^n .$$

Below is a table of values of the probability for a given m and n.

$m\backslash n$	10	15	20	25	30	50	100
10	0.9590	0.9917	0.9983	0.9997	0.9999	1.	1.
100	0.5675	0.7155	0.8129	0.8770	0.9191	0.9849	0.9998
1000	0.2257	0.3186	0.4004	0.4724	0.5357	0.7216	0.9225
10000	0.0770	0.1132	0.1481	0.1815	0.2136	0.3301	0.5512
100000	0.0249	0.0372	0.0493	0.0612	0.0730	0.1187	0.2232
1000000	0.0080	0.0119	0.0158	0.0198	0.0237	0.0391	0.0767

Table 1. Probability (rounded) that among n people someone flips a coin at least m times before they get the same number of Heads as Tails. For example, for a class of 30 students there is better than even chance that at least 1000 flips will be required for at least one student in order for the game to terminate.

The Birthday paradox is the surprising statement that you only need 23 people in a room so that the probability of two sharing the same birthday is at least 50%. We can adapt our coin flip game to this setting. In Table 2, we show the number of students n needed to have a better than even chance that one of them will flip the coin at least m times.

Therefore, from Table 2, we can see that you only need 9 students to make it more likely than not that one student will need to flip the coin at least 100 hundred times to get the same number of Heads as Tails.

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Summary. We consider a probability question and use it to revisit two famous paradoxes, the St. Petersburg paradox and the Birthday paradox.

m	n
10	3
100	9
1000	28
10000	87
100000	275
1000000	869

Table 2. Number of students n that guarantee a greater than even chance that at least one of them flips a coin at least m times. For example, with 275 students there is a greater than even chance that one of them will have to flip a coin at least 100000 times to get the same number of Heads as Tails.

References

1. Robert J Aumann, *The St. Petersburg paradox: A discussion of some recent comments*, Journal of Economic Theory **14** (1977), no. 2, 443 – 445.
2. Daniel Bernoulli, *Exposition of a new theory on the measurement of risk*, Econometrica **22** (1954), no. 1, 23–36.
3. Allan J. Ceasar, *A Monte Carlo simulation related to the St. Petersburg paradox*, The College Mathematics Journal **15** (1984), no. 4, 339–342.
4. William Feller, *An introduction to probability theory and its applications. Vol. I*, Third edition, John Wiley & Sons, Inc., New York-London-Sydney, 1968.
5. Charles M. Grinstead and J. Laurie Snell, *Introduction to probability*, Second edition.
6. Dominic Klyve and Anna Lauren, *An empirical approach to the St. Petersburg paradox*, College Math. J. **42** (2011), no. 4, 260–264.
7. Karl Menger, *Das Unsicherheitsmoment in der Wertlehre*, Zeitschrift für Nationalökonomie **5** (1934), no. 4, 459–485.
8. R. Monte, *The random walk for dummies*, MIT Undergraduate Journal of Mathematics **1** (1999), no. 1, 143–148.
9. Georg Pólya, *Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz*, Math. Ann. **84** (1921), no. 1-2, 149–160.
10. Lloyd S Shapley, *The St. Petersburg paradox: A con game?*, Journal of Economic Theory **14** (1977), no. 2, 439 – 442.
11. Richard P. Stanley and and, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999,