A Short Proof of a Sum of Powers Formula

We give a combinatorial proof of the following classical result:

Theorem.

$$1^k + 2^k + \cdots + n^k = \sum_{j=1}^{k} j! \left\{ {k \atop j} \right\} \binom{n+1}{j+1}, \tag{1}$$

where $\left\{ {n \atop k} \right\}$*, denoting Stirling numbers of the second kind, is the number of ways of partitioning a set of* n *labelled objects into* k *nonempty unlabelled subsets.*

Proof. Observe that

$$\sum_{i=1}^{n} i^k = \sum_{i \le n} \sum_{a_1 \le i} \sum_{a_2 \le i} \cdots \sum_{a_k \le i} 1 = \sum_{1 \le a_1, a_2, \ldots, a_k \le i \le n} 1.$$

Therefore, we need to count how many $k+1$ tuples $(a_1, a_2, \ldots, a_k, i)$ exist with the constraints $1 \le a_m \le i \le n$ for all $1 \le m \le k$. To simplify, make the change of variable $\ell = i + 1$ and count the tuples $(a_1, a_2, \ldots, a_k, \ell)$ such that $1 \le a_m < \ell \le n + 1$.

Let $j \in \{1, \ldots, k\}$. Consider a subset S of $\{1, 2, \ldots, n+1\}$ with $j+1$ elements. We want to count the number of tuples $(a_1, a_2, \ldots, a_k, \ell)$ satisfying $1 \le a_m < \ell \le n+1$ and that the values taken by the elements in the tuple are exactly those in S. Since $\ell > a_m$ for all m, we see that ℓ is forced to be the maximum among the elements of S. Then we have the other j possible values from S distributed among $a_1, a_2, \ldots, a_k$. There are $\left\{ {k \atop j} \right\}$ ways of partitioning $\{a_1, a_2, \ldots a_k\}$ into j blocks.[1] Then there are $j!$ ways of assigning values to those blocks from the values left in S. Therefore we have $j!\left\{ {k \atop j} \right\}$ ways of matching the tuples to the values of S. There are $\binom{n+1}{j+1}$ ways of choosing S. After summing over j we recover (1). ■

Remark: The proof shares similarities with [**1**] where the author also translates the sum of powers into the same combinatorial problem as we do. However, the subsequent techniques used are different.

REFERENCES

1. G. Mackiw, A combinatorial approach to sums of integer powers, *Math. Magazine* **73** no. 1 (2000) 44–46.

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[1] To clarify what we mean by blocks, consider the following example: In the $k = 3$ case with $j = 2$, we want 2 blocks. There are 3, namely $\{\{a_1, a_2\}, \{a_3\}\}$, $\{\{a_1, a_3\}, \{a_2\}\}$, and $\{\{a_2, a_3\}, \{a_1\}\}$.

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