A direct combinatorial approach to the sum of the kth powers of the first n integers

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Abstract

We translate the sum $1^k + 2^k + \ldots + n^k$ into a combinatorial problem. We then prove a classic formula using this technique.

Many papers regarding finding formulas for $S_k = 1^k + 2^k + 3^k + \ldots + n^k$ have been published in MAA journals. We have geometric approaches to the problem (see [2] and [4]), we have algebraic approaches that use the binomial formula on $(n+1)^k - n^k$ and telescope (see [6] and [10]), we have combinatorial attacks that count lattice points in different ways (see [5] and [7]) and we have attacks using linear algebra (see [9]). The classical way to attack the problem is using the telescoping technique together with generating functions (see page 160 in [8]).

Our goal is to explain an approach that we consider more direct and easier to remember. In this paper we will prove the following classic formula that uses Stirling numbers of the second kind[1]:

Theorem 1.

$$S_k = 1^k + 2^k + \ldots + n^k = \sum_{j=1}^{k} j! \left\{ {k \atop j} \right\} \binom{n+1}{j+1}, \tag{1}$$

where $\left\{ {n \atop k} \right\}$*, denoting Stirling numbers of the second kind, is the number of ways of partitioning a set of* n *labelled objects into* k *nonempty unlabelled subsets.*

We will first show the easy case of the sum of the first n positive integers. Start by transforming the sum into a double sum as follows:

$$\sum_{i=1}^{n} i = \sum_{i \le n} \sum_{j \le i} 1 = \sum_{j \le i \le n} 1.$$

Therefore $1 + 2 + \ldots + n$ is the number of pairs (j, i) satisfying that $1 \le j \le i \le n$. We translated the sum into a combinatorial problem. We can consider all pairs $j < i \le n$ and all pairs $j = i \le n$. Therefore

$$1 + 2 + \ldots + n = \binom{n}{2} + \binom{n}{1} = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

This proof of the sum of $1 + 2 + \ldots + n$ can be easily generalized. To make the process clearer for generalization let's consider the sum of the first n squares:

$$\sum_{c \le n} c^2 = \sum_{c \le n} \sum_{a \le c} \sum_{b \le c} 1 = \sum_{1 \le a, b \le c \le n} 1.$$

[1]Another important formula regarding S_k is Faulhaber's formula (see [1] and [3]).

That is, we want to find the number of triples (a, b, c) satisfying that $1 \leq a \leq c$ and $1 \leq b \leq c$. We can split this into the following cases: $(a < b < c), (a < b = c), (a = b < c), (a = b = c)$. By symmetry of a and b we can just multiply the cases of the form $(a < b)$ by 2. Then, we have

$$\sum_{c=1}^{n} c^2 = 2\binom{n}{3} + 2\binom{n}{2} + \binom{n}{2} + \binom{n}{1} = 2\binom{n}{3} + 3\binom{n}{2} + \binom{n}{1}$$
$$= \frac{n}{6}\left(2(n-1)(n-2) + 9(n-1) + 6\right) = \frac{n(n+1)(2n+1)}{6}.$$

In general, the technique breaks down the sum of the first n k-powers into a sum of $k+1$ terms as follows:

Lemma 1.

$$S_k = \sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1} c_j \binom{n}{j}, \tag{2}$$

where $c_j = j!\left\{{k \atop j}\right\} + (j-1)!\left\{{k \atop j-1}\right\}$.

Proof. Using the same idea as the examples for $k = 1, 2$, we start by translating the sum of powers into a $(k+1)$-fold sum:

$$\sum_{i=1}^{n} i^k = \sum_{i \leq n} \sum_{a_1 \leq i} \sum_{a_2 \leq i} \sum_{a_3 \leq i} \cdots \sum_{a_k \leq i} 1 = \sum_{1 \leq a_1, a_2, \ldots, a_k \leq i \leq n} 1.$$

Therefore, we translated the problem to the combinatorial problem of counting how many $k+1$ tuples $(a_1, a_2, \ldots, a_k, i)$ exist with the constraints $1 \leq a_m \leq i \leq n$ for all $1 \leq m \leq k$.

Let $j \in \{1, 2, \ldots, n\}$. Consider a subset S of $\{1, 2, \ldots, n\}$ with j elements. We want to count the number of tuples $(a_1, a_2, \ldots, a_k, i)$ satisfying $1 \leq a_m \leq i \leq n$ and that the values taken by the elements in the tuple are exactly those in S. Since $i \geq a_m$ for all m, then i is forced to be the maximum among the elements of S. Now we have two cases: $a_m < i$ for all m or there exists an m such that $a_m = i$.

- If $a_m < i$ for all m, then we have $j-1$ possible values from S distributed among $a_1, a_2, \ldots a_k$. There are $\left\{{k \atop j-1}\right\}$ of partitioning the $a_1, a_2, \ldots a_k$ into $j-1$ "blocks"[2]. Then there are $(j-1)!$ ways of assigning values to those "blocks" from the values left in S. Therefore we have $(j-1)!\left\{{k \atop j-1}\right\}$ ways of matching the tuples to the values of S.

- If there is an m such that $a_m = i$, then we have j possible values from S distributed among the $a_1, a_2, \ldots a_k$. Therefore, we have $j!\left\{{k \atop j}\right\}$ ways of matching the tuples to the values of S.

Since there are $\binom{n}{j}$ possible subsets S, then the total number of tuples is (2), and the proof is complete. □

Now, we can easily prove the main theorem:

[2] To clarify what we mean by "blocks", let's look at the following example: in the $k = 3$ case when $j = 3$, we want 2 "blocks", there are 3 ways to do this, namely $\{\{a_1, a_2\}, \{a_3\}\}, \{\{a_1, a_3\}, \{a_2\}\}$, and $\{\{a_2, a_3\}, \{a_1\}\}$.

Proof of Theorem 1. From Lemma 1 we get

$$\begin{aligned}\sum_{i=1}^{n} i^k &= \sum_{j=1}^{k+1}\left(j!\left\{{k \atop j}\right\} + (j-1)!\left\{{k \atop j-1}\right\}\right)\binom{n}{j}\\ &= \sum_{j=1}^{k+1} j!\left\{{k \atop j}\right\}\binom{n}{j} + \sum_{j=1}^{k+1}(j-1)!\left\{{k \atop j-1}\right\}\binom{n}{j}.\end{aligned}$$

In the left sum the $j = k+1$ is zero because $\left\{{k \atop k+1}\right\} = 0$, and we can make a change of variable in the right sum to get

$$\begin{aligned}\sum_{i=1}^{n} i^k &= \sum_{j=1}^{k} j!\left\{{k \atop j}\right\}\binom{n}{j} + \sum_{j=1}^{k} j!\left\{{k \atop j}\right\}\binom{n}{j+1}\\ &= \sum_{j=1}^{k} j!\left\{{k \atop j}\right\}\binom{n+1}{j+1}.\end{aligned}$$

For the last step we used Pascal's formula for binomial coefficients. □

References

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