Swimming Lanes Problem

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Consider the following puzzle from the Riddler column on the FiveThirtyEight website:

Its summertime and my local swimming pool, which has exactly five swimming lanes (and no general swim area), may be opening in the coming weeks. It remains unclear what social distancing practices will be required, but its quite possible that swimmers will not be allowed to occupy adjacent lanes. Under these guidelines, the pool could accommodate at most three swimmers one each in the first, third and fifth lanes. Suppose a queue of swimmers arrives at the pool when it opens at 9 a.m. One at a time, each person randomly picks a lane from among the lanes that are available (i.e., the lane has no swimmer already and is not adjacent to any lanes with swimmers), until no more lanes are available. At this point, what is the expected number of swimmers in the pool? Extra credit: Instead of five lanes, suppose there are N lanes. When no more lanes are available, what is the expected number of swimmers in the pool?

If we number the lanes 1,2,3,4,5. We have the following situations:

- If the first swimmer takes lanes 2 or 4, then there will be exactly 2 swimmers. This contributes $2 \cdot \frac{2}{5}$ to the expected value.
- If the first swimmer takes lane 3, then the next swimmer is forces 1 or 5, leaving the other one open for swimmer three. Therefore, there will be exactly 3 swimmers. This contributes $3 \cdot \frac{1}{5}$ to the expected value.
- If the first swimmer takes lanes 1 or 5, we can assume by symmetry that it will be lane 1. Then if the second swimmer takes lanes 3 or 5, there will be three swimmers total, and if the second swimmer takes lane 4, there will be 2 swimmers. Therefore, this contributes $\frac{2}{5}\left(\frac{1}{3}2 + \frac{2}{3}3\right)$ to the expected value.

Adding everything up we get that the expected value is $\frac{37}{15}$.

Now, let's attack the general case. Let $a(n)$ be the expected value with n lanes. Suppose the first swimmer takes lane k. Then it splits the pool into $(k-2)$ lanes on the left and $(n-k-1)$ on the right where we can place swimmers. (Note: If $k=1$ or $k=n$ then you'll get -1 lanes on one side, we'll simply define $a(-1)=0$ to account for this.) Then

$$a(n) = \sum_{k=1}^{n} \frac{1}{n}(1 + a(k-2) + a(n-k-1)) = 1 + \frac{2}{n}\sum_{k=1}^{n-2} a(k).$$

Now, note that

$$a(n) - a(n-1) = \frac{2}{n}a(n-2) - \frac{2}{n(n-1)}\sum_{k=1}^{n-3} a(k).$$

But

$$a(n-1)-1=\frac{2}{n-1}\sum_{k=1}^{n-3}a(k),$$

so

$$a(n)-a(n-1)=\frac{2}{n}a(n-2)-\frac{1}{n}(a(n-1)-1).$$

Therefore, we have the easier recursion

$$a(n)=a(n-1)\left(1-\frac{1}{n}\right)+\frac{2}{n}a(n-2)+\frac{1}{n}.$$

Let's consider the generating function

$$A(x)=\sum_{n=1}^{\infty}a(n)x^n.$$

Then

$$\begin{aligned}
A(x) &= x+x^2+\sum_{n=3}^{\infty}a(n)x^n \\
&= x+x^2+\sum_{n=3}^{\infty}a(n-1)x^n-\sum_{n=3}^{\infty}\frac{a(n-1)}{n}x^n+2\sum_{n=3}^{\infty}\frac{a(n-2)}{n}x^n+\sum_{n=3}^{\infty}\frac{x^n}{n} \\
&= x+x^2+x\sum_{n=2}^{\infty}a(n)x^n-\sum_{n=3}^{\infty}\frac{a(n-1)}{n}x^n+2\sum_{n=3}^{\infty}\frac{a(n-2)}{n}x^n+\sum_{n=3}^{\infty}\frac{x^n}{n} \\
&= x+x^2+x(A(x)-x)-\sum_{n=3}^{\infty}\frac{a(n-1)}{n}x^n+2\sum_{n=3}^{\infty}\frac{a(n-2)}{n}x^n+\sum_{n=3}^{\infty}\frac{x^n}{n}
\end{aligned}$$

Therefore

$$A(x)=x+xA(x)-\sum_{n=3}^{\infty}\frac{a(n-1)}{n}x^n+2\sum_{n=3}^{\infty}\frac{a(n-2)}{n}x^n+\sum_{n=3}^{\infty}\frac{x^n}{n}.$$

Now take a derivative with respect to x

$$\begin{aligned}
A'(x) &= 1+A(x)+xA'(x)-\sum_{n=3}^{\infty}a(n-1)x^{n-1}+2\sum_{n=3}^{\infty}a(n-2)x^{n-1}+\sum_{n=3}^{\infty}x^{n-1} \\
&= 1+A(x)+xA'(x)-(A(x)-x)+2xA(x)+\frac{x^2}{1-x} \\
&= \frac{1}{1-x}+2xA(x)+xA'(x).
\end{aligned}$$

After clearing the denominator $(1-x)$, if we let $y=A(x)$ we have the differential equation

$$(1+2x(1-x)y)-(1-x)^2y'=0.$$

Let $M(x,y)=1+2x(1-x)y$ and $N(x,y)=-(1-x)^2$. Then

$$\frac{M_y-N_x}{N}=\frac{2x(1-x)-2(1-x)}{-(1-x)^2}=2.$$

Therefore, we can multiply the differential equation by e^{2x} to make it *exact*:

$$e^{2x}\left(1+2x(1-x)y\right)+(-e^{2x}(1-x)^2)y'=0.$$

Let Ψ be such that $\Psi_x = e^{2x}\left(1+2x(1-x)y\right)$ and $\Psi_y = (-e^{2x}(1-x)^2)$. Then

$$\Psi(x,y) = -e^{2x}(1-x)^2y + f(x) = -e^{2x} + 2xe^{2x} - x^2e^{2x} + f(x).$$

Then

$$\begin{aligned}\Psi_x(x,y) &= -2e^{2x} + 2e^{2x} + 4xe^{2x} - 2xe^{2x} - 2x^2e^{2x} + f'(x)\\ &= 2xe^{2x} - 2x^2e^{2x} + f'(x) = 2x(1-x)e^{2x} + f'(x).\end{aligned}$$

Therefore $f'(x) = e^{2x}$, therefore $f(x) = \frac{e^{2x}}{2}$.

That means that the solution of the differential equation must satisfy $\Psi(x,y) = c$, i.e.,

$$-e^{2x}(1-x)^2y + \frac{e^{2x}}{2} = c.$$

For $x = 0, y = 0$, therefore $c = 1/2$. Finally, we can solve for y and get

$$A(x) = y = \frac{\frac{1}{2} - \frac{e^{2x}}{2}}{-e^{2x}(1-x)^2} = \frac{1-e^{-2x}}{2(1-x)^2}.$$

Now that we have $A(x)$ we can figure out a formula for $a(n)$. First note

$$\begin{aligned}1-e^{-2x} &= -\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}x^n,\\ \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty}(n+1)x^n.\end{aligned}$$

Therefore

$$A(x) = -\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}x^n\right)\left(\sum_{n=0}^{\infty}(n+1)x^n\right) = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{(-2)^k}{k!}(n+1-k)\right)x^n.$$

Therefore

$$\begin{aligned}a(n) &= -\frac{1}{2}\sum_{k=1}^{n}\frac{(-2)^k}{k!}(n+1-k)\\ &= -\frac{n+1}{2}\sum_{k=1}^{n}\frac{(-2)^k}{k!} - \sum_{k=1}^{n}\frac{(-2)^{k-1}}{(k-1)!}\\ &= -\left(\frac{n+1}{2}\right)\left(\sum_{k=1}^{\infty}\frac{(-2)^k}{k!} - \sum_{k=n+1}^{\infty}\frac{(-2)^k}{k!}\right) - \sum_{k=0}^{\infty}\frac{(-2)^k}{k!} + \sum_{k=n}^{\infty}\frac{(-2)^k}{k!}.\end{aligned}$$

Using that

$$e^{-2} = \sum_{k=0}^{\infty}\frac{(-2)^k}{k!},$$

we get

$$a(n) = -\left(\frac{n+1}{2}\right)\left(e^{-2}-1\right) + \left(\frac{n+1}{2}\right)\sum_{k=n+1}^{\infty}\frac{(-2)^k}{k!} - e^{-2} + \sum_{k=n}^{\infty}\frac{(-2)^k}{k!}$$
$$= \left(\frac{1-e^{-2}}{2}\right)n + \frac{1-3e^{-2}}{2} + \left(\frac{n+1}{2}\right)\sum_{k=n+1}^{\infty}\frac{(-2)^k}{k!} + \sum_{k=n}^{\infty}\frac{(-2)^k}{k!}.$$

Since $\frac{2^k}{k!}$ is a decreasing function for $k \geq 1$ and $\sum_{k=m}^{\infty}\frac{(-2)^k}{k!}$ is an alternating series, then

$$\left|\sum_{k=m}^{\infty}\frac{(-2)^k}{k!}\right| \leq \frac{2^m}{m!}.$$

Therefore

$$\left|\left(\frac{n+1}{2}\right)\sum_{k=n+1}^{\infty}\frac{(-2)^k}{k!}\right| \leq \left(\frac{n+1}{2}\right)\frac{2^{n+1}}{(n+1)!} = \frac{2^n}{n!}.$$

Therefore

$$a(n) = \left(\frac{1-e^{-2}}{2}\right)n + \frac{1-3e^{-2}}{2} + \theta_n,$$

where

$$-\frac{2^{n+1}}{n!} \leq \theta_n \leq \frac{2^{n+1}}{n!}.$$

$\theta_n \to 0$ quite fast, so we have an excellent approximation of $a(n)$. Furthermore, we can see that the proportion of swimmers in the lanes goes to $(1-e^{-2})/2$ as $n \to \infty$.