Introduction

A triangular number is a number \( N \) that satisfies that \( N \) dots can be arranged in increasing order to form an equilateral triangle as in the figure below:

![The first few triangular numbers](Image 57x2308 to 288x2535)

Figure 1: The first few triangular numbers: 1, 3, 6, and 10.

These numbers are of the form:

\[
\frac{n(n+1)}{2}
\]

There happen to be triangular numbers that, when added to consecutive triangular numbers, create another triangular number, and this can be represented by an equation.

For \( k \) consecutive triangular numbers that add up to be another triangular number, where \( k, m, \) and \( n \) are positive integers, we have:

\[
\frac{n(n+1)}{2} + \ldots + \frac{(n+k-1)(n+k)}{2} = \frac{(m)(m+1)}{2}
\]

With some work the equation can be rearranged conveniently into the form \( (x)^2 - k(y)^2 = f \) like this:

\[
(2m+1)^2 - k(2n+1)^2 = k^2 - 4k + 3
\]

We’ll explore for what \( k \) can we always find positive integers \( m \) and \( n \). In particular, we’ll prove:

\[
\text{Theorem}
\]

Let \( k > 4 \) be a square. Then there exist \( k \) consecutive triangular numbers that add up to a triangular number.

Example and Preliminary Work

Let’s look at an example of when \( k = 4 \).

\[
(2m + 1)^2 - 4(2n + 1)^2 = (4^1 - 16 + 3)
\]

Use difference of squares for the left hand side:

\[
(x)^2 - (ay)^2 = (x + ay)(x - ay), \text{ so:}
\]

\[
(2m + 1 - 4n - 8)(2m + 1 + 4n + 8) = 17.
\]

The integer divisors on the left hand side are 1, 17, -1, and -17, which gives:

\[
(m, n) = (4, 0), (4, -4), (-5, 0), \text{ and } (-5, -4).
\]

Let’s see if \( k = a^2 > 4 \) we have good solutions:

\[
(2m + 1)^2 - a^2(2n + 1)^2 = \frac{a^6 - 4a^2 + 3}{3}.
\]

LHS : \((2m + 1 + 2na + a^3)(2m + 1 - 2na - a^3)\)

RHS : \(\frac{(a + 1)(a - 1)(a^2 + a^2 - 3)}{3}\)

Where the left hand sides factors are \( d' \) and \( d \).

When \( k \) is even

Let’s look at the proof for the even case, when the square is even.

One solid set of divisors is 1 and \( f \) for any case so we can see that when the equations are added and solved for \( m \), we can get:

\[
m = \frac{d + d' - 2}{4} \quad \text{and} \quad \frac{d + d' + 2}{4}
\]

Similarly when one is subtracted from the other and solved for \( n \), we get:

\[
n = \frac{d' - d - 2a^3}{4a}
\]

Therefore we have the solutions:

\[
m = \frac{(a^4 - 4)(a^2)}{12}
\]

\[
n = \frac{a(a^2 - 6a - 4)}{12}
\]

When \( k \) is odd

When \( a \equiv 1 \pmod{3} \), \( f \) can be split up into integer divisors:

\[
f = \frac{(a - 1)(a + 1)}{6}(a^4 - a^2 - 3)
\]

So we have divisors:

\[
m - an + \frac{1 - a^3}{2} = \frac{a + 1}{2} = d
\]

and

\[
m + an + \frac{1 + a^3}{2} = \frac{a - 1}{6}(a^4 + a^2 - 3) = d'
\]

When they are added or subtracted from one another we can solve for \( m \) and \( n \):

\[
m = \frac{a^2(a - 1)(a^2 + 1)}{12}
\]

\[
n = \frac{(a + 2)(a - 3)(a^2 + 1)}{12}
\]

An analogous process can be used for \( a \equiv 2 \) and 0 (mod 3).

Further Studies

- How many solutions are there for a particular \( k \)?
- Can we prove that there are always positive integers \( m \) and \( n \) for any \( k \neq 4 \) consecutive triangular numbers?

References


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