Playing with Triangular Numbers

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Introduction

A triangular number is a number N that satisfies that N dots can be arranged in increasing order to form an equilateral triangle as in the figure below.

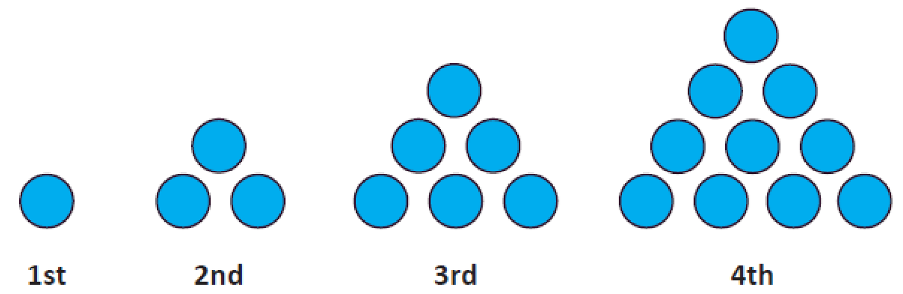

Figure 1: The first few triangular numbers: 1, 3, 6, and 10.

These numbers are of the form:

$$\frac{n(n+1)}{2}$$

There happen to be triangular numbers that, when added to consecutive triangular numbers, create another triangular number, and this can be represented by an equation.
For k consecutive triangular numbers that add up to be another triangular number, where k, m, and n are positive integers, we have:

$$\frac{n(n+1)}{2} + \ldots + \frac{(n+k-1)(n+k)}{2} = \frac{(m)(m+1)}{2}.$$

With some work the equation can be rearranged conveniently into the form $(x)^2 - k(y)^2 = f$ like this:

$$(2m+1)^2 - k(2n+k)^2 = \frac{k^3 - 4k + 3}{3}.$$

We'll explore for what k can we always find positive integers m and n. In particular, we'll prove:

Theorem

Let $k > 4$ be a square. Then there exist k consecutive triangular numbers that add up to a triangular number.

Example and Preliminary Work

Let's look at an example of when $k = 4$.

$$(2m+1)^2 - 4(2n+4)^2 = \frac{(4^3 - 16 + 3)}{3}.$$

Use difference of squares for the left hand side: $(x)^2 - (ay)^2 = (x+ay)(x-ay)$, so:

$$(2m+1-4n-8)(2m+1+4n+8) = 17.$$

The integer divisors on the left hand side are 1, 17, -1, and -17, which gives:

$$(m,n) = (4,0), (4,-4), (-5,0), \text{ and } (-5,-4).$$

Let's see if $k = a^2 > 4$ we have good solutions:

$$(2m+1)^2 - a^2(2n+a^2)^2 = \frac{a^6 - 4a^2 + 3}{3}.$$

$$\text{LHS}: (2m+1+2na+a^3)(2m+1-2na-a^3)$$

$$\text{RHS}: \frac{(a+1)(a-1)(a^4+a^2-3)}{3}$$

Where the left hand sides factors are d' and d.

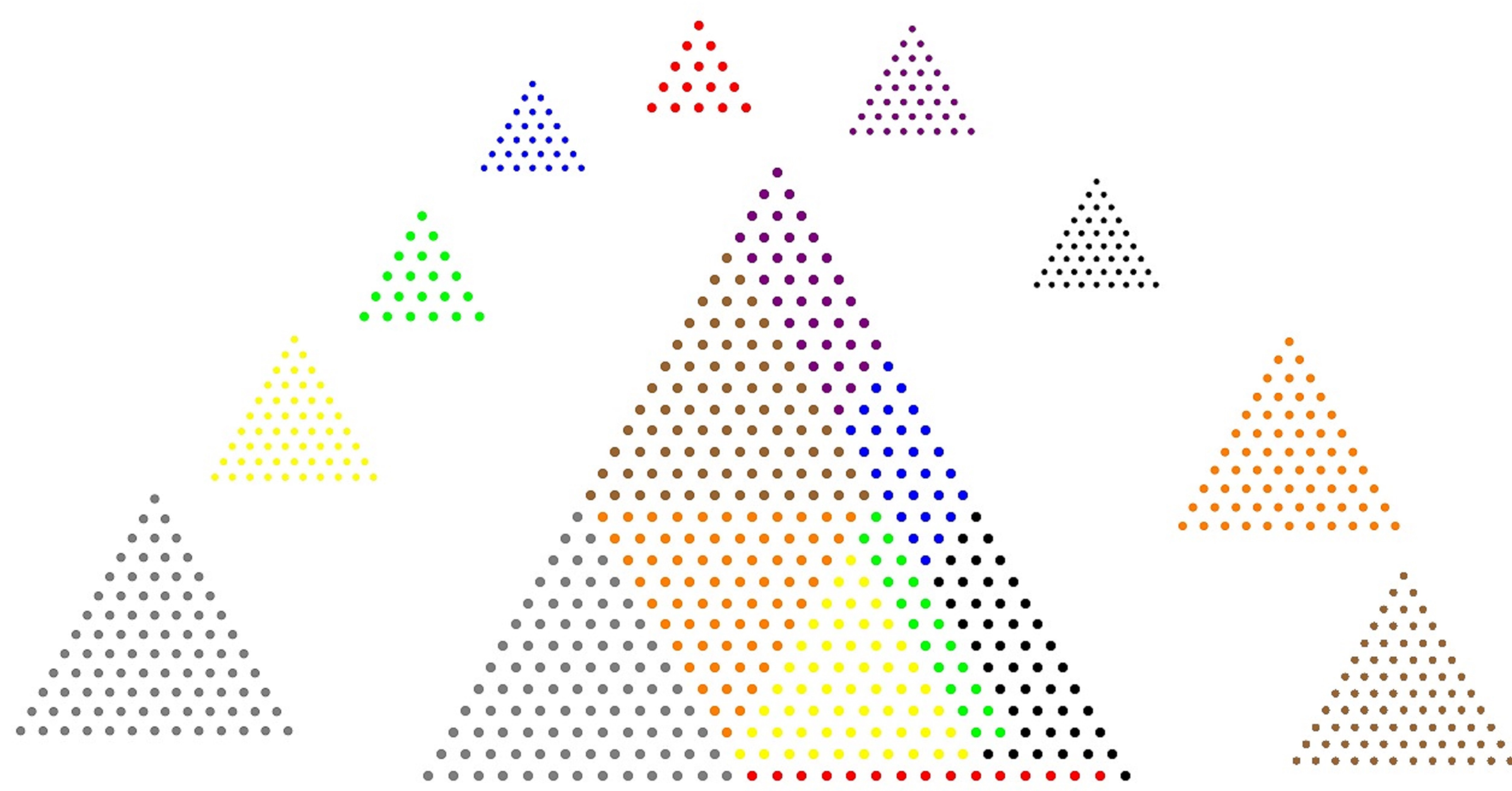

Figure 2: The sum of 9 consecutive triangular numbers, starting from the 5th triangular number, makes the 29th triangular number.

When k is even

Let's look at the proof for the even case, when the square is even.
One solid set of divisors is 1 and f for any case so we can see see that when the equations are added and solved for m, we can get:

$$m = \frac{d + d' - 2}{4} = \frac{1 + \frac{(a-1)(a+1)(a^4+a^2-3)}{3} - 2}{4}$$

Similarly when one is subtracted from the other and solved for n, we get:

$$n = \frac{d' - d - 2a^3}{4a} = \frac{\frac{(a-1)(a+1)(a^4+a^2-3)}{3} - 1 - 2a^3}{4a}$$

Therefore we have the solutions:

$$m = \frac{(a^4-4)(a^2)}{12}$$
$$n = \frac{a(a^4 - 6a - 4)}{12}$$

When k is odd

When $a \equiv 1 \pmod 3$, f can be split up into integer divisors:

$$f = \left(\frac{a-1}{6}\right)\left(\frac{a+1}{2}\right)(a^4 + a^2 - 3)$$

So we have divisors:

$$m - an + \frac{1-a^3}{2} = \frac{a+1}{2} = d$$

and

$$m + an + \frac{1+a^3}{2} = \frac{a-1}{6}(a^4 + a^2 - 3) = d'$$

When they are added together or subtracted from one another we can solve for m and n:

$$m = \frac{a^2(a-1)(a^2+1)}{12}$$
$$n = \frac{(a+2)(a-3)(a^2+1)}{12}$$

An analogous process can be used for $a \equiv 2$ and $0 \pmod 3$.

Further Studies

- How many solutions are there for a particular k?
- Can we prove that there are always positive integers m and n for any $k \neq 4$ consecutive triangular numbers?

References

1. *Triangular Numbers.* Nzmaths. Accessed June 06, 2018. https://nzmaths.co.nz/resource/triangular-numbers.
2. Matthew McMullen, *Playing with blocks*, Math Horiz. **25** (2018), no. 4, 14–15.

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