Report on the 14th Annual USA Junior Mathematical Olympiad

Béla Bajnok and Enrique Treviño

Photo

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The USA Junior Mathematical Olympiad (USAJMO) is the final round in the American Mathematics Competitions series for high school students in grade 10 or below, organized each year by the Mathematical Association of America. The competition follows the style of the International Mathematical Olympiad (IMO): it consists of three problems each on two consecutive days, with an allowed time of four and a half hours both days.

The 14th annual USAJMO was given on Tuesday, March 21, 2023 and Wednesday, March 22, 2023, and was taken by 273 students. The names of the winners and those receiving honorable mention, as well as more information on the American Mathematics Competitions program, can be found on the site https://www.maa.org/math-competitions. Below we present the problems and solutions of the competition; a similar article for the USA Mathematical Olympiad (USAMO), offered to high school students in grade 12 or below, can be found in a concurrent issue of *Mathematics Magazine*.

The problems of the USAJMO are chosen—from a large collection of proposals submitted for this purpose—by the USAMO/USAJMO Editorial Board that works under the leadership of co-editors-in-chief Oleksandr Rudenko and Enrique Treviño. This year's problems were created by Titu Andreescu, Holden Mui, Carl Schildkraut, David Torres, and Anton Trygub.

The solutions presented here are those of the present authors, relying in part on the submissions of the problem authors. Each problem was worth 7 points; the nine-tuple

$$(n; a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$$

states the number of students who submitted a paper for the relevant problem, followed by the numbers who scored $7, 6, \ldots, 0$ points, respectively.

Problem 1 $(261; 182, 8, 10, 5, 2, 4, 12, 38)$; *proposed by Titu Andreescu.* Find all triples of positive integers (x, y, z) that satisfy the equation

$$2(x + y + z + 2xyz)^2 = (2xy + 2yz + 2zx + 1)^2 + 2023. \tag{1}$$

Solution. Expanding expressions, we get

$$2(x+y+z+2xyz)^2 = 2x^2+2y^2+2z^2+8x^2y^2z^2+4xy+4xz+4yz+8x^2yz+8xy^2z+8xyz^2,$$

and

$$(2xy+2yz+2zx+1)^2 = 4x^2y^2+4y^2z^2+4x^2z^2+1+8x^2yz+8xy^2z+8xyz^2+4xy+4xz+4yz.$$

After cancellations and the rearrangements of terms, we thus see that our original equation (1) is equivalent to

$$8x^2y^2z^2 - 4x^2y^2 - 4y^2z^2 - 4x^2z^2 + 2x^2 + 2y^2 + 2z^2 - 1 = 2023. \tag{2}$$

Our goal is to factor the left side, and for that it is helpful to observe that if $2x^2 = 1$, or $2y^2 = 1$, or $2z^2 = 1$, then that side vanishes. This allows us to rewrite (2) as

$$(2x^2 - 1)(2y^2 - 1)(2z^2 - 1) = 2023.$$

Therefore, $2x^2 - 1$, $2y^2 - 1$, and $2z^2 - 1$ must be positive divisors of 2023 that multiply to 2023. Now $2023 = 7 \cdot 17^2$, so 2023 has six positive divisors: 1, 7, 17, 119, 289, and 2023. Adding 1 and dividing by 2 results in a square number only in the first three cases, which then means that $2x^2-1$, $2y^2 - 1$, and $2z^2 - 1$ must equal 7, 17, and 17 in some order, yielding three choices for (x, y, z): $(2, 3, 3)$, $(3, 2, 3)$, and $(3, 3, 2)$.

Problem 2 $(245; 146, 0, 2, 0, 1, 5, 18, 73)$; *proposed by Holden Mui.* In an acute triangle ABC, let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of $\overline{AQ}$. Prove that $NB = NC$.

First solution. We will prove that N lies on the perpendicular bisector of BC, from which the claim follows. Let T be the intersection point of AQ with the perpendicular bisector of BC; we need to prove that $T = N$.

Applying the Law of Sines to triangle ATM, we have

$$AT = AM \cdot \frac{\sin(\angle AMT)}{\sin(\angle ATM)}. \tag{3}$$

Now

$$\angle AMT = 90^\circ - \angle BMA = 90^\circ - \angle CMP = \angle PCM,$$

so

$$\sin(\angle AMT) = \sin(\angle PCM) = \frac{PM}{MC} = \frac{PM}{BM}.$$

Furthermore, $\angle ATM$ and $\angle MTQ$ are supplementary angles, so

$$\sin(\angle ATM) = \sin(\angle MTQ) = \frac{MQ}{TQ}.$$

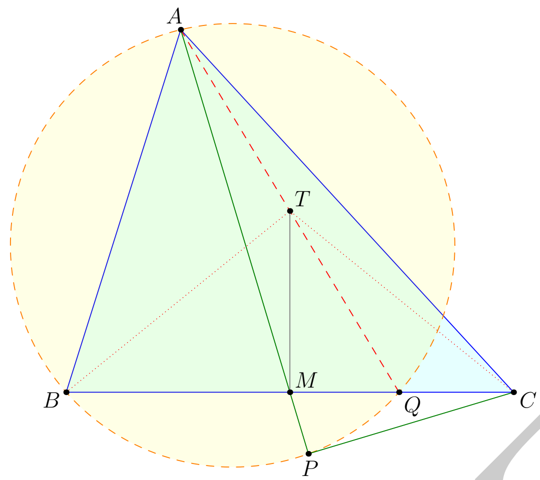

Figure 1: Illustration for the first solution to Problem 2.

Therefore, we may rewrite (3) as

$$AT = \frac{AM \cdot PM \cdot TQ}{BM \cdot MQ}.$$

Since $\angle QAM = \angle QAP = \angle QBP = \angle MBP$ and $\angle AMQ = \angle PMB$, triangles AMQ and BMP are similar, and thus $AM/MQ = BM/PM$. This gives $AT = TQ$, as claimed.

Second solution. We let ω_1 be the circumcircle of triangle ABP, and ω_2 be the circumcircle of triangle APC. The foot of the perpendicular from A to BC, denoted by D, is then on ω_2, since $\angle ADC = \angle APC$.

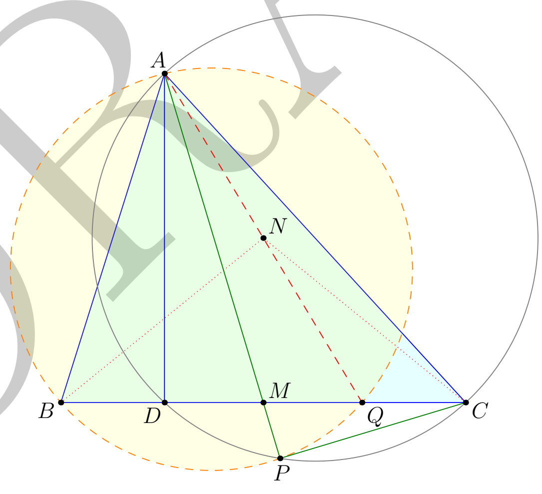

Figure 2: Illustration for the second solution to Problem 2.

From the Power of a Point Theorem applied to circles ω_1 and ω_2, respectively, we get

$$MA \cdot MP = MB \cdot MQ$$

and

$$MA \cdot MP = MC \cdot MD;$$

since $MB = MC$, we get $MD = MQ$. But then MN is a midline of triangle ADQ, and thus MN and AD are parallel. Therefore, N lies on the perpendicular bisector of BC, which implies our claim.

Problem 3 $(233; 37, 10, 20, 19, 60, 31, 19, 37)$; *proposed by Holden Mui.* Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a *maximal grid-aligned configuration* on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes do not overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find the maximum value of $k(C)$ as a function of n.

Solution. The answer is $(n+1)^2/4$.

We first prove that $k(C)$ is at most $(n+1)^2/4$. We label the squares of the board by ordered pairs (i, j) in the usual manner, and then color each square by one of three colors: if both of its coordinates are odd, we color it *red*; if both of its coordinates are even, we color it *blue*; and if its coordinates have different parities, we color it *white*. Note that we then have exactly $(n+1)^2/4$ red squares, $(n-1)^2/4$ blue squares (together, $(n^2+1)/2$ *dark* squares), and $(n^2-1)/2$ white squares.

In a maximal grid-aligned configuration, each domino covers a dark square and a white square. We will refer to each domino by the color of the dark square it covers. We will call the square that is not covered by dominoes *empty*. Note that, by parity, the empty square is colored dark.

Let Γ be the directed graph whose vertices are the dark squares, and whose directed edges are drawn from a square v to the square that the domino covering v points to (the square that, if it were empty, the domino could slide to in one move), if it exists. An example is shown in Figure 3. Note that Γ uniquely determines the configuration of dominoes.

Let G be the undirected graph corresponding to Γ. Observe that the connected components of G are formed by monochromatic dominoes (i.e., either all red or all blue). Suppose that C is a cycle in G. Connecting the center points of the unit squares in C forms a polygon P. Since each side of P has even length, the region surrounded by P can be divided into 2×2 squares, so the area of the region is divisible by 4, and is thus even. By Pick's Theorem, the area equals $B/2 + I - 1$, where B is the number of unit squares in C, and I is the number of unit squares in the interior of P. Since B is twice the number of dominoes in C, and C contains an even number of dominoes, we find that I is odd, which can only be true if C encloses the empty square (i.e., contains it in its interior).

Let T be the connected component of G that contains the empty square u; according to what we just proved, T cannot contain a cycle, so it must be a tree. Let $\Gamma(T)$ be the subgraph of Γ on T. Since T is a tree, it is valid to say whether the edges in $\Gamma(T)$ point towards u or away from it. We claim that, in fact, each directed edge in $\Gamma(T)$ points towards u. Suppose indirectly that there are some directed edges that point away from u, and choose one whose tail vertex v_1 is closest to u; let v_2 be the head of this directed edge. Let v_3 be the vertex in T that v_1 is adjacent to along the path from v_1 to u. By assumption, the edge connecting v_1 and v_3 has v_1 as its tail. But v_2 and v_3 are different vertices, so v_1 is the tail of more than one directed edge, which is a contradiction.

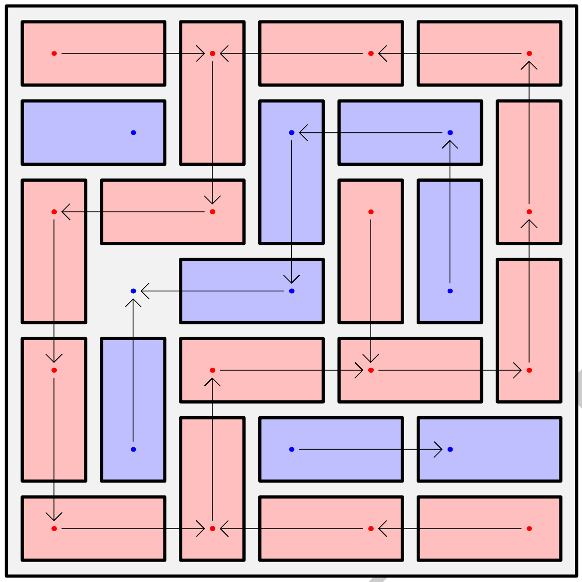

Figure 3: Illustration for a maximal grid-aligned configuration and its corresponding directed graph.

This means that u is the universal sink of $\Gamma(T)$. We thus find that there is exactly one way to make any vertex of $\Gamma(T)$ become the empty square: reverse the direction of each edge on the path that connects that vertex to u, that is, slide each corresponding domino on that path towards u. Therefore, $k(C)$ equals the number of vertices in T. Since T is monochromatic, it can have at most as many vertices as there are blue squares or red squares, which implies that $k(C)$ is at most $(n+1)^2/4$. We can achieve $k(C) = (n+1)^2/4$ by positioning the dominoes covering red squares in a snake-like fashion. An example construction for $n = 7$ is shown in Figure 4.

Problem 4 $(254; 26, 10, 11, 22, 28, 95, 43, 19)$; *proposed by David Torres.* Two players, B and R, play the following game on an infinite grid of unit squares, all initially colored white. The players take turns starting with B. On B's turn, B selects one white unit square and colors it blue. On R's turn, R selects two white unit squares and colors them red. The players alternate until B decides to end the game. At this point, B gets a score, given by the number of unit squares in the largest (in terms of area) simple polygon containing only blue unit squares. What is the largest score B can guarantee? (A *simple polygon* is a polygon (not necessarily convex) that does not intersect itself and has no holes.)

Solution. We claim that the maximum score that B may achieve is 4.

It is not hard to see that B can always guarantee a score of 4. Indeed, during the first four rounds, she can always pick a white square neighboring—and by that we mean, having a common side with—one of the squares she already picked: at least two of the four squares neighboring her first choice are still white, at least two of the six squares neighboring her first two squares are still white, and at least one of the seven or eight squares neighboring her first three squares are still white.

To show that R can prevent B from obtaining a score larger than 4, we divide the infinite grid into 2×2 squares that we call *states*. (Figure 5 illustrates this with thicker lines marking state

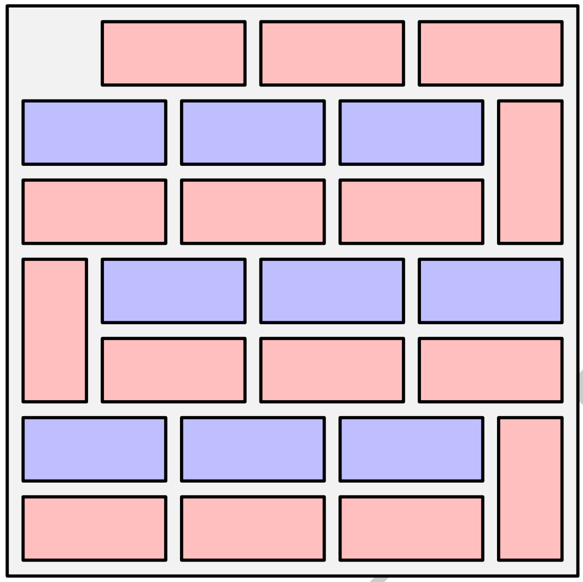

Figure 4: A maximal grid-aligned configuration achieving $k(C) = (n+1)^2/4$.

borders.)

Note that each square now has exactly two neighboring squares that are in other states. The strategy for R is then to choose the two squares that border B's latest choice across state lines (unless one or both of these squares are already colored, in which case R can instead pick any other square or squares).

With this strategy, R is able to prevent B from ever coloring two neighboring squares blue that belong to different states. But this means that B can never form a simple polygon containing only blue unit squares that contains squares from more than one state. This proves our claim.

Problem 5 $(234; 103, 7, 6, 2, 11, 6, 55, 44)$; *proposed by Carl Schildkraut.* Positive integers a and N are fixed, and N positive integers are written on a blackboard. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n+a$, and on Bob's turn, he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends. After analyzing the N integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these N integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

Solution. If there is only a single integer on the board, then all moves are determined uniquely, and thus there is nothing to prove. Below we suppose that there are at least two numbers on the board.

Before we make a precise claim regarding the game, let us recall that every positive integer m can be written uniquely as $m = 2^k \cdot c$ where c is odd; k is called the 2-*adic valuation* of m and is denoted by $\nu_2(m)$. Here, for our fixed value of a, we say that a positive integer m is 2-*perfect* if $\nu_2(m) = \nu_2(a)$, 2-*abundant* if $\nu_2(m) > \nu_2(a)$, and 2-*deficient* if $\nu_2(m) < \nu_2(a)$. We can now make the following claims: If the board has at least one number that is 2-perfect or 2-abundant, then

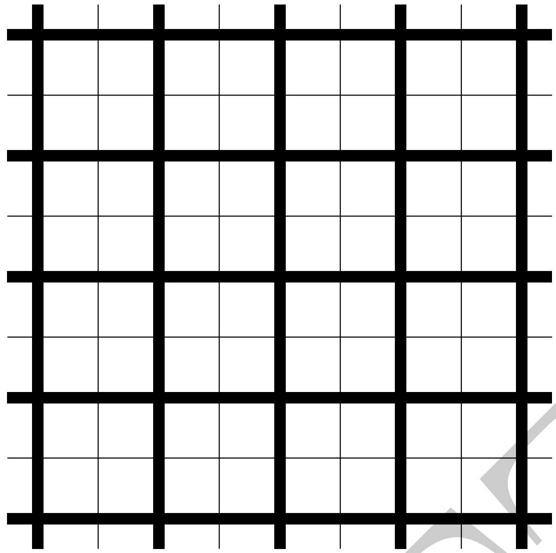

Figure 5: An illustration for Problem 4.

Alice can prolong the game indefinitely, but if all numbers on the board are 2-deficient, then the game will terminate (in a finite number of steps) no matter how Alice and Bob play.

For our proof, it is helpful to make the following observations about any positive integer m:

- if m is 2-perfect, then $\nu_2(m+a) > \nu_2(m)$, and thus $m+a$ is 2-abundant;
- if m is 2-abundant, then $\nu_2(m+a) = \nu_2(a)$, and thus $m+a$ is 2-perfect;
- if m is 2-deficient, then $\nu_2(m+a) = \nu_2(m)$, and thus $m+a$ remains 2-deficient.

To verify the first observation, let $m = 2^k \cdot c_1$ and $a = 2^k \cdot c_2$ for some nonnegative integer k and odd integers c_1 and c_2. Then

$$m + a = 2^k \cdot (c_1 + c_2),$$

where c_1+c_2 is even and hence $\nu_2(m+a) > k$. The other two observations can be deduced similarly.

Now we turn to the proof of our claims. Assume first that the board contains at least one number that is 2-perfect or 2-abundant; we claim that Alice can then move in such a way that after her move the board contains at least one 2-abundant number. Indeed, if there is already a 2-abundant number on the board, she should leave it unchanged and add a to any other number, and if there is a 2-perfect one, she can add a to it to make it 2-abundant. (Note that Alice may have several such options, in which case she can choose any of them.) After Alice's move, Bob will have some moves available (for example, on the 2-abundant number on the board). He will reduce the 2-adic valuation of one of the numbers on the board by exactly 1, but that will transfer the scenario back to a case in which the board contains at least one number that is 2-perfect or 2-abundant. Alice can thus use this strategy to prolong the game indefinitely.

Suppose now that all numbers on the board are 2-deficient. We let $S = \sum \nu_2(n)$, where the summation is taken over all integers on the board. By our observations above, every move that Alice makes will leave S unchanged, while obviously each move that Bob makes reduces S by 1.

But then, after a finite number of rounds, there will be no more even numbers on the board, and the game terminates.

Problem 6 $(182; 6, 0, 1, 0, 4, 0, 20, 151)$; *proposed by Anton Trygub.* Isosceles triangle ABC, with $AB = AC$, is inscribed in circle ω. Let D be an arbitrary point on $\overline{BC}$ such that $BD \neq DC$. Ray AD intersects ω again at E (other than A). Point F (other than E) is chosen on ω such that $\angle DFE = 90°$. Line FE intersects rays AB and AC at points X and Y, respectively. Prove that $\angle XDE = \angle EDY$.

Solution. We introduce three additional points: we let G be the intersection of FD with ω (other than F), I be the intersection of AG and CE, and J be the intersection of AG and BE. Then, since $AB = AC$, we have

$$\angle JEA = \angle BEA = \angle AEC = \angle AEI,$$

and since we also have $\angle EAG = \angle EFG = 90°$, this means that $\overline{EA}$ is both an angle bisector and an altitude of triangle EIJ. But D lies on EA, so we have $\angle JDA = \angle ADI$ and thus

$$\angle JDE = \angle EDI. \tag{4}$$

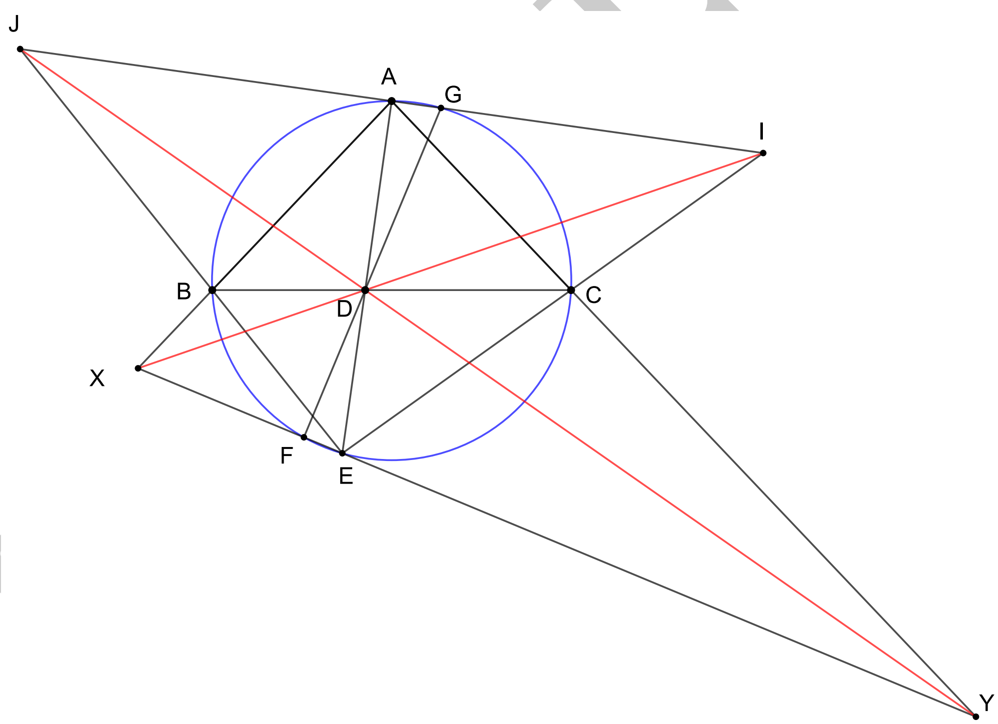

Figure 6: Illustration for the solution to Problem 6.

To complete our proof, we will employ *Pascal's Theorem*, which states that for a (not necessarily convex) cyclic hexagon $A_1A_2A_3A_4A_5A_6$, the intersections of the opposite sides are collinear; that is, if $L = A_1A_2 \cap A_4A_5$, $M = A_2A_3 \cap A_5A_6$, and $N = A_3A_4 \cap A_6A_1$, then L, M, and N are collinear.

Now applying Pascal's theorem to the cyclic hexagon $ABCEFG$, since $X = AB \cap EF$, $D = BC \cap FG$, and $I = CE \cap GA$, we have that X, D, and I are collinear. Likewise, in the cyclic hexagon $ACBEFG$, since $Y = AC \cap EF$, $D = CB \cap FG$, and $J = BE \cap GA$, we have that Y, D, and J are collinear. Therefore, we have $\angle XDJ = \angle YDI$, and this together with (4) implies that $\angle XDE = \angle YDE$, as claimed.

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Summary. We present the problems and solutions to the 14th Annual United States of America Junior Mathematical Olympiad.

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