Report on the 15th Annual USA Junior Mathematical Olympiad

Béla Bajnok, Evan Chen, and Enrique Treviño

Photo

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Photo

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The USA Junior Mathematical Olympiad (USAJMO) is the final round in the American Mathematics Competitions series for high school students in grade 10 or below, organized each year by the Mathematical Association of America. The competition follows the style of the International Mathematical Olympiad (IMO): it consists of three problems each on two consecutive days, with an allowed time of four and a half hours both days.

The 15th annual USAJMO was given on Tuesday, March 19 and Wednesday, March 20, 2024, and was taken by 261 students. The names of the winners and those receiving honorable mention, as well as more information on the American Mathematics Competitions program, can be found on the site https://maa.org/student-programs/amc/. Below we present the problems and solutions of the competition; a similar article for the USA Mathematical Olympiad (USAMO), offered to high school students in grade 12 or below, can be found in a concurrent issue of *Mathematics Magazine*.

The problems of the USAJMO are chosen—from a large collection of proposals submitted for this purpose—by the USAMO/USAJMO Editorial Board that works under the leadership of co-editors-in-chief Oleksandr Rudenko and Enrique Treviño. This year's problems were created by

Serena An, John Berman, Evan O'Dorney, Carl Schildkraut, Alec Sun, Anton Trygub, and Claire Zhang.

The solutions presented here are those of the present authors, relying in part on the submissions of the problem authors. Each problem was worth 7 points; the nine-tuple

$$(n; a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$$

states the number of students who submitted a paper for the relevant problem, followed by the numbers who scored $7, 6, \ldots, 0$ points, respectively.

Problem 1 $(251; 157, 43, 4, 1, 0, 3, 11, 32)$; *proposed by Evan O'Dorney.* Let $ABCD$ be a cyclic quadrilateral with $AB = 7$ and $CD = 8$. Points P and Q are selected on line segment AB so that $AP = BQ = 3$. Points R and S are selected on line segment CD so that $CR = DS = 2$. Prove that $PQRS$ is a cyclic quadrilateral.

Solution. Let ρ and O denote the radius and center of the circle circumscribing $ABCD$, respectively. We will show that, in fact, P, Q, R, S lie on a circle centered at O; that is, the lengths PO, QO, RO, SO are all equal.

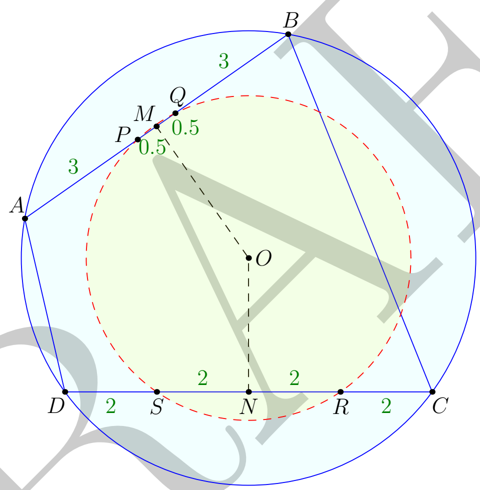

Figure 1: Illustration for Problem 1.

Our proof will use repeated applications of the Pythagorean Theorem. Let M denote the midpoint of PQ, which is also the midpoint of AB, as $AM = MB = 3.5$ and $PM = QM = 0.5$. Assuming M is distinct from O, it follows OM is the perpendicular bisector of AB, and so from the Pythagorean Theorem we get

$$PO^2 = PM^2 + MO^2 = PM^2 + (AO^2 - AM^2) = 0.5^2 + \rho^2 - 3.5^2$$
$$QO^2 = QM^2 + MO^2 = QM^2 + (BO^2 - BM^2) = 0.5^2 + \rho^2 - 3.5^2.$$

Note that the same equations are valid when $O = M$.

Similarly, let N denote the midpoint of RS, which is also the midpoint of CD, as $CN = DN = 4$ and $RN = SN = 2$. Repeating the same calculation gives

$$RO^2 = RN^2 + NO^2 = RN^2 + (CO^2 - CN^2) = 2^2 + \rho^2 - 4^2$$
$$SO^2 = SN^2 + NO^2 = SN^2 + (DO^2 - DN^2) = 2^2 + \rho^2 - 4^2.$$

From this it follows that

$$PO^2 = QO^2 = RO^2 = SO^2 = \rho^2 - 12,$$

proving our claim.

Problem 2 $(229; 57, 10, 0, 0, 0, 43, 9, 110)$; *proposed by Serena An and Claire Zhang.* Let m and n be positive integers. Let S be the set of lattice points (x, y) with $1 \le x \le 2m$ and $1 \le y \le 2n$. A configuration of mn axis-parallel rectangles is called *happy* if each point of S is the vertex of exactly one rectangle. Prove that the number of happy configurations is odd.

Solution. Let us denote by $f(2m, 2n)$ the number of happy configurations for a $2m \times 2n$ grid of lattice points; the problem then asks us to show that $f(2m, 2n)$ is always odd.

We start by evaluating $f(2, 2n)$. Note that the top row is the top edge of some rectangle, and there are $2n - 1$ choices for the bottom edge of that rectangle. For each such choice, we can take out those two rows to derive the recurrence

$$f(2, 2n) = (2n - 1) \cdot f(2, 2n - 2),$$

at which point repeating the argument again gives

$$f(2, 2n) = (2n - 1) \cdot (2n - 3) \cdot f(2, 2n - 4),$$

and so on. Since $f(2, 2) = 1$, we get

$$f(2, 2n) = (2n - 1)!! = (2n - 1) \cdot (2n - 3) \cdot \cdots \cdot 3 \cdot 1, \tag{1}$$

which is odd.

Let us now consider the general situation. We will separate the $f(2m, 2n)$ happy configurations into two groups, as follows. Given a happy configuration $\mathcal{C}$, we let $\tau(\mathcal{C})$ be the configuration obtained by swapping the last two columns. An illustration for $m = 3$ and $n = 2$ is provided in Figure 2. Note that $\tau(\mathcal{C})$ is also a happy configuration, and τ is an involution: $\tau(\tau(\mathcal{C})) = \mathcal{C}$ for every happy $\mathcal{C}$. We then say that $\mathcal{C}$ is *single* if $\tau(\mathcal{C}) = \mathcal{C}$; otherwise we say that $\mathcal{C}$ and $\tau(\mathcal{C})$ are *partnered* with each other.

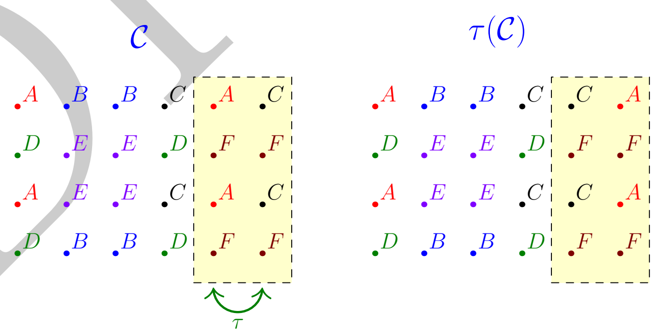

Figure 2: A happy configuration $\mathcal{C}$ with its image $\tau(\mathcal{C})$.

Since partnered happy configurations come in pairs, the number $p(2m, 2n)$ of partnered happy configurations is even. Single happy configurations can be described readily: they are the ones whose last two columns are self-contained, meaning that every rectangle with a vertex in these columns is completely contained in these two columns. Hence it follows that

$$f(2m, 2n) = p(2m, 2n) + f(2m-2, 2n) \cdot f(2, 2n);$$

since $p(2m, 2n)$ is even and $f(2, 2n)$ is odd, we find that $f(2m, 2n)$ and $f(2m-2, 2n)$ have the same parity for all $m > 1$ and $n \geq 1$. Applying this procedure recursively yields that $f(2m, 2n)$ and $f(2, 2n)$ have the same parity, and this proves that $f(2m, 2n)$ is odd.

Problem 3 $(213; 24, 1, 1, 0, 1, 21, 6, 159)$; *proposed by John Berman.* A sequence $a_1, a_2, \ldots$ of positive integers is defined recursively by $a_1 = 2$ and $a_{n+1} = a_n^{n+1} - 1$ for $n \geq 1$. Prove that for every odd prime p and positive integer k, some term of the sequence is divisible by p^k.

Solution. With a given odd prime p and a positive integer k, let

$$n = p^k - 2p^{k-1} + 1.$$

We will prove that a_{n-1} or a_{n+2} is divisible by p^k. More specifically, we will establish the following two claims:

(i) If a_n is divisible by p, then a_{n+2} is divisible by p^k.

(ii) If a_n is not divisible by p, then a_{n-1} is divisible by p^k.

To prove (i), note that if a_n is divisible by p, then a_n^{n+1} is divisible by p^{n+1}. But $a_n^{n+1} = a_{n+1} + 1$ and $k \leq n+1$, so $a_{n+1} + 1$ is divisible by p^k. Since $n+2$ is even, $a_{n+2} = a_{n+1}^{n+2} - 1$ is divisible by $a_{n+1} + 1$, and hence by p^k, as claimed.

Turning to our second claim, we assume that a_n is not divisible by p, and first prove that a_{n-1} is divisible by p. Indeed, if this were not the case, then by Fermat's Little Theorem we would have $a_{n-1}^{p-1} \equiv 1 \bmod p$. Since n is divisible by $p-1$, this would further imply that $a_{n-1}^n \equiv 1 \bmod p$, meaning that $a_n = a_{n-1}^n - 1$ is divisible by p, contradicting our assumption.

Next, we prove that $a_{n-2} \equiv 1 \bmod p$. To see that, note that $a_{n-2}^{n-1} = a_{n-1} + 1 \equiv 1 \bmod p$. As a_{n-2} is not divisible by p, Fermat's Little Theorem yields $a_{n-2}^{p-1} \equiv 1 \bmod p$; since n is divisible by $p-1$, this implies that $a_{n-2}^n \equiv 1 \bmod p$. So $a_{n-2}^n \equiv 1 \equiv a_{n-2}^{n-1} \bmod p$, and we get $a_{n-2} \equiv 1 \bmod p$, as claimed.

We now use induction on i to prove that $a_{n-2}^{p^{i-1}} \equiv 1 \bmod p^i$ for all $i \in \mathbb{N}$. The case $i = 1$ was just established, so we may assume that $a_{n-2}^{p^{i-2}} \equiv 1 \bmod p^{i-1}$ for some $i \geq 2$. Therefore, we have an integer c for which

$$a_{n-2}^{p^{i-2}} = cp^{i-1} + 1,$$

and we get

$$a_{n-2}^{p^{i-1}} = (cp^{i-1} + 1)^p = 1 + \binom{p}{1} cp^{i-1} + \binom{p}{2} c^2 p^{2i-2} + \cdots + c^p p^{pi-p}.$$

Since each term on the right except for the first is divisible by p^i, our claim follows.

Finally, note that $n-1$ is divisible by p^{k-1}, so from the previous claim we get that $a_{n-2}^{n-1} \equiv 1 \bmod p^k$, and thus $a_{n-1} = a_{n-2}^{n-1} - 1$ is divisible by p^k, proving claim (ii).

Note: The assumption that p is odd is necessary: for $p = 2$ the sequence alternates between integers congruent to 2 mod 4 and ones that are 3 mod 4, so no term will be divisible by 2^k for any $k > 1$.

Problem 4 $(248; 119, 11, 19, 0, 16, 37, 3, 43)$; *proposed by Alec Sun.* Let $n \geq 3$ be an integer. Rowan and Colin play a game on an $n \times n$ grid of squares, where each square is colored either red or blue. Rowan is allowed to permute the rows of the grid and Colin is allowed to permute the columns. A grid coloring is *orderly* if:

- no matter how Rowan permutes the rows of the coloring, Colin can then permute the columns to restore the original grid coloring; and
- no matter how Colin permutes the columns of the coloring, Rowan can then permute the rows to restore the original grid coloring.

In terms of n, how many orderly colorings are there?

Solution. We start by giving some examples of orderly colorings:

- the all-blue coloring,
- the all-red coloring,
- each of the $n!$ colorings where every row and column has exactly one red cell, and
- each of the $n!$ colorings where every row and column has exactly one blue cell.

These $2n! + 2$ colorings are orderly. We will now show that these are the only ordering colorings and hence the answer to the question is $2n! + 2$.

Let $\mathcal{A}$ be any orderly coloring. Consider any particular column C in $\mathcal{A}$, and let m denote the number of red cells that C has. Any row permutation σ that Rowan chooses will transform C into some column $\sigma(C)$, and our assumption requires $\sigma(C)$ to appear as a column somewhere in the original assignment $\mathcal{A}$. Therefore, the columns of the grid must contain all $\binom{n}{m}$ possible patterns with m red cells. Since $\binom{n}{m} > n$ when $2 \leq m \leq n-2$, we must have $m \in \{0, 1, n-1, n\}$.

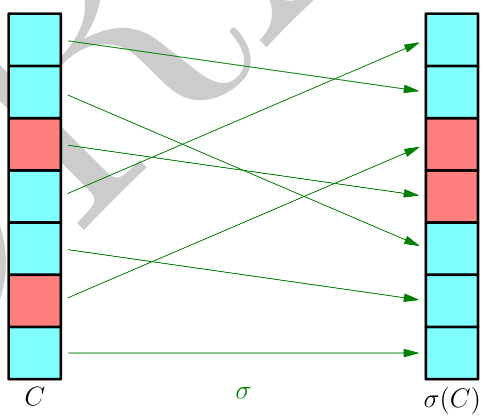

Figure 3: Illustration for Problem 4 with $n = 7$, $m = 2$, and a permutation σ.

Moreover, if either $m = 1$ or $m = n - 1$, then these columns use up all n columns of the grid; this leads to the $2n!$ cases in the third and fourth bullets above.

The only remaining case is when $m \in \{0, n\}$ for every column; that is, every column is monochromatic. In that case, Rowan's operations have no effect on the grid. Therefore, for the coloring to be orderly, Colin's operations must not change the grid either, and hence all columns are the same color, yielding our first or second bullet above. The proof is now complete.

Problem 5 $(235; 50, 12, 1, 1, 15, 14, 71, 71)$; *proposed by Carl Schildkraut.* Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x^2 - y) + 2yf(x) = f(f(x)) + f(y) \tag{2}$$

for all $x, y \in \mathbb{R}$.

Solution. We will prove that there are exactly three solutions: $f(x) = 0$, $f(x) = x^2$, and $f(x) = -x^2$. It is easy to verify that these functions satisfy the equation; we will prove that there are no others. Our strategy will be to prove that the function is even, and then use that to find $f(x)$ in terms of $f(1)$.

First note that by setting $y = 0$, (2) becomes

$$f(x^2) = f(f(x)) + f(0). \tag{3}$$

Subtracting (3) from (2) yields

$$f(x^2 - y) + 2yf(x) + f(0) = f(x^2) + f(y), \tag{4}$$

and then substituting x with $-x$, we get

$$f(x^2 - y) + 2yf(-x) + f(0) = f(x^2) + f(y).$$

Since our last two equations must hold for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we get $f(x) = f(-x)$, so f is even.

Now, substituting y with y^2 in (4) gives us

$$f(x^2 - y^2) + 2y^2 f(x) + f(0) = f(x^2) + f(y^2). \tag{5}$$

Since f is even, $f(x^2 - y^2) = f(y^2 - x^2)$, so replacing (x, y) with (y, x) in (5) yields

$$f(x^2 - y^2) + 2x^2 f(y) + f(0) = f(x^2) + f(y^2). \tag{6}$$

From (5) and (6) we deduce

$$y^2 f(x) = x^2 f(y). \tag{7}$$

Let $f(1) = k$. Setting $y = 1$ in (7) yields $f(x) = kx^2$ for all $x \in \mathbb{R}$. Now (2) translates to

$$k(x^2 - y)^2 + 2kyx^2 = k^3x^4 + ky^2,$$

and thus

$$(k^3 - k)x^4 = 0.$$

Therefore $k = 0,\ 1,\ -1$, and the three solutions are $f(x) = 0$, $f(x) = x^2$, and $f(x) = -x^2$, as claimed.

Problem 6 $(182; 1, 0, 0, 0, 0, 0, 1, 180)$; *proposed by Anton Trygub.* Point D is selected inside acute triangle ABC so that $\angle DAC = \angle ACB$ and $\angle BDC = 90^\circ + \angle BAC$. Point E is chosen on ray BD

so that $AE = EC$. Let M be the midpoint of BC. Show that line AB is tangent to the circumcircle of triangle BEM.

Solution. As $EA = EC$, we know E lies on the perpendicular bisector of AC. We reflect point B across this perpendicular bisector to a point Q, and we reflect B across E to a point F. Note that this gives an isosceles (and hence cyclic) trapezoid $ABQC$, and that lines QF and AC are perpendicular.

We deal with the configuration shown in Figure 4, where we assume $AB < BC < AC$, so that B, A, D lie on one side of the perpendicular bisector of AC, and F, Q, M, C lie on the other side. The proofs of other cases are handled analogously (alternatively, experts may prefer to take directed angles modulo 180° to avoid the case distinction).

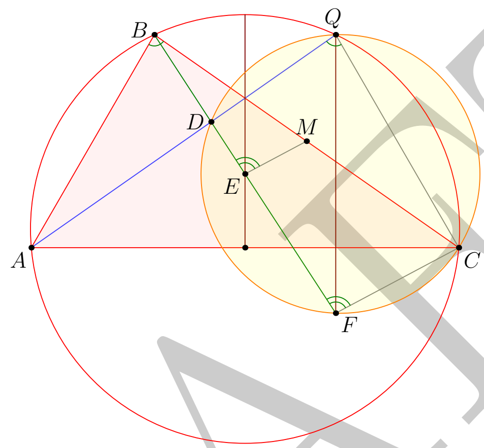

Figure 4: Illustration for Problem 6.

The condition that $\angle DAC = \angle ACB$ then implies that D lies on ray AQ, and the condition that $\angle BDC = 90° + \angle BAC$ implies that

$$\angle FDC = 180° - \angle BDC = 90° - \angle BAC = 90° - \angle QCA = \angle FQC.$$

Therefore, the four points F, D, Q, C lie on a circle ω, and thus $\angle DFC$ and $\angle DQC$ are supplementary angles.

Since E and M are the midpoints of BF and BC, respectively, we also know that lines EM and FC are parallel. Therefore,

$$\angle BEM = \angle BFC = \angle DFC = 180° - \angle DQC.$$

But

$$\angle DQC = \angle AQC = \angle ABC = \angle ABM,$$

so $\angle BEM = 180° - \angle ABM$.

To complete our proof, consider the tangent line ℓ to the circumcircle of triangle BEM at B. Note that the angle between ℓ and BM equals the inscribed angle $\angle BEM$ in the circle. But this means that ℓ must coincide with line AB, as claimed.

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Summary. We present the problems and solutions to the 15th Annual United States of America Junior Mathematical Olympiad.

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