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Report on the 53rd Annual USA Mathematical Olympiad

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The USA Mathematical Olympiad (USAMO) is the final round in the American Mathematics Competitions series for high school students, organized each year by the Mathematical Association of America. The competition follows the style of the International Mathematical Olympiad (IMO): it consists of three problems each on two consecutive days, with an allowed time of four and a half hours both days.

The 53rd annual USAMO was given on Tuesday, March 19 and Wednesday, March 20, 2024. Of the 286 students taking the exam, 17, 34, and 48 earned Gold, Silver, and Bronze Prizes, respectively; an additional 42 students received Honorable Mention. The names of the prize winners, as well as more information on the American Mathematics Competitions program, can be found on the site https://maa.org/student-programs/amc/. Below we present the problems and solutions of the competition; a similar article for the USA Junior Mathematical Olympiad (USAJMO), offered to students in grade 10 or below, can be found in a concurrent issue of the *College Mathematics Journal*.

The problems of the USAMO are chosen—from a large collection of proposals submitted for this purpose—by the USAMO/USAJMO Editorial Board, under the leadership of co-editors-in-chief Oleksandr Rudenko and Enrique Treviño. This year's problems were created by Titu Andreescu, Krit Boonsiriseth, Rishabh Das, Gabriel Dospinescu, Luke Robitaille, and Anton Trygub.

The solutions presented here are those of the present authors, relying in part on the submissions of the problem authors. Each problem was worth 7 points; the nine-tuple

$$(n; a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$$

states the number of students who submitted a paper for the relevant problem, followed by the numbers who scored $7, 6, \ldots, 0$ points, respectively.

Problem 1 $(283; 153, 25, 36, 7, 5, 10, 23, 24)$; *proposed by Luke Robitaille*. Find all integers $n \geq 3$ such that the following property holds: if we list the divisors of $n!$ in increasing order as $1 = d_1 < d_2 < \cdots < d_k = n!$, then we have

$$d_2 - d_1 \leq d_3 - d_2 \leq \cdots \leq d_k - d_{k-1}.$$

First solution. We can easily check that $n = 3$ and $n = 4$ have the required property, as the divisors of 3! are $1, 2, 3, 6$, and the divisors of 4! are $1, 2, 3, 4, 6, 8, 12, 24$. We will prove that no value of $n \geq 5$ satisfies the property; in fact, we show that if N is a multiple of 60 (as $n!$ is for $n \geq 5$), then its divisors cannot satisfy the chain of inequalities needed.

Let N be a multiple of 60, q be the smallest positive integer not dividing N, and let d be the smallest divisor of N greater than q; we then have

$$7 \leq q < d \leq N.$$

Note that $q-2$, $q-1$, and d are consecutive divisors of N, and thus so are N/d, $N/(q-1)$, and $N/(q-2)$. If N were to have the required property, then we would have

$$\frac{N}{q-1} - \frac{N}{d} \leq \frac{N}{q-2} - \frac{N}{q-1};$$

since $d \geq q+1$, this would imply that

$$\frac{N}{q-1} - \frac{N}{q+1} \leq \frac{N}{q-2} - \frac{N}{q-1}.$$

Since this inequality simplifies to $q \leq 5$, we arrived at a contradiction, establishing our claim.

Second solution. As above, we can check that $n = 3$ and $n = 4$ satisfy the required property; we can also verify that $n = 5$ and $n = 6$ do not: 15, 20, and 24 are consecutive divisors of 5! (but $20 - 15 > 24 - 20$), and 6, 8, and 9 are consecutive divisors of 6! (but $8 - 6 > 9 - 8$). We will prove that no value of $n \geq 7$ has the required property either.

We will rely on Bertrand's Postulate (also known as Chebyshev's Theorem), which states that for every integer $n > 1$, there is a prime p with $n < p < 2n$.[1] Note that p is not a divisor of $n!$, and hence $n!$ must have a pair of consecutive divisors below $2n$ that are more than 1 apart. Our task is then accomplished if we exhibit two divisors of $n!$ that are greater than $2n$ and are 1 apart.

In the case when $n \geq 7$ is odd, $n^2 - 2n = (n-2)n$ and $n^2 - 2n + 1 = 2 \cdot \frac{n-1}{2} \cdot (n-1)$ are both divisors of $n!$ (note that $2 < \frac{n-1}{2}$), are 1 apart, and are greater than $2n$. Similarly, when $n \geq 8$ is even, $n^2 - 4n + 3 = (n-3)(n-1)$ and $n^2 - 4n + 4 = 2 \cdot \frac{n-2}{2} \cdot (n-2)$ are both divisors of $n!$, are 1 apart, and are greater than $2n$. This completes our proof.

Problem 2 $(248; 20, 12, 3, 3, 3, 3, 25, 36, 146)$; *proposed by Rishabh Das*. Let $S_1, S_2, \ldots, S_{100}$ be finite sets of integers whose intersection is not empty. For each non-empty $T \subseteq \{S_1, S_2, \ldots, S_{100}\}$, the size of the intersection of the sets in T is a multiple of the number of sets in T. What is the least possible number of elements that are in at least 50 sets?

Solution. The answer is $50\binom{100}{50}$.

We introduce some notation, as follows. For a collection $U = \{S_1, S_2 \ldots, S_{100}\}$ of finite sets of integers and an integer a, we define the *support* $C(a)$ *of* a *in* U as the

[1] The statement was conjectured by Bertrand [**1**] in 1845, and first proved by Chebyshev [**5**] in 1852. Erdős [**3**] gave a beautiful elementary proof of the result in 1932, and was fond of mentioning "Chebyshev said it, and I say it again, there is always a prime between n and $2n$."

collection of sets $S \in U$ that contain a:

$$C(a) = \{S \in U \mid a \in S\}.$$

For each subset A of U, we then let $f(A)$ denote the number of integers whose support is A:

$$f(A) = \big|\{a \in \mathbb{Z} \mid C(a) = A\}\big|.$$

Observe that for any subset T of U, the size of the intersection of the sets in T is given by $\sum_{T \subseteq A} f(A)$. Therefore, our task is to find the minimum possible value of $F = \sum_{|T| \geq 50} f(T)$, so that U is a collection of 100 finite sets of integers whose intersection is not empty and that the *divisibility requirement* is satisfied, that is, $F(T) = \sum_{T \subseteq A} f(A)$ is divisible by $|T|$ for any $T \subseteq U$.

We now describe an operation on our sets of integers that preserves the divisibility requirement and does not change the value of F. Suppose that there is a collection $A = \{S_{i_1}, S_{i_2}, \dots, S_{i_{|A|}}\} \subseteq U$ with $f(A) \geq |A|$. We can then find pairwise distinct integers $a_1, a_2, \dots, a_{|A|}$ so that $S_{i_j} \in C(a_j)$ for $j = 1, 2, \dots, |A|$. We then define the *push-down* A' of A as the result of removing these elements from their respective sets, that is, we set

$$A' = \{S_{i_1} \setminus \{a_1\}, S_{i_2} \setminus \{a_2\}, \dots, S_{i_{|A|}} \setminus \{a_{|A|}\}\}.$$

Observe that when $|A| \geq 51$, then a push-down on A preserves both the value of $F = \sum_{|T| \geq 50} f(T)$ and the divisibility requirement. Indeed, we have $f(A') = f(A) - |A|$, but $f(T)$ increases by 1 for every $T \subset A$ with $|T| = |A| - 1$, while remaining unaffected for other sets T. Therefore, F is unchanged, and $F(T)$ is either unchanged or decreased by $|T|$.

Let A be a maximal subset of U for which $f(A) > 0$; that is, $f(A) > 0$ but $f(B) = 0$ for all $A \subset B$. Then $F(A) = f(A)$, so by the divisibility requirement, $f(A)$ is divisible by $|A|$, which implies that $f(A) \geq |A|$. If $|A| \geq 51$, we apply a push-down on A to get A'. After repeated applications of this process, we arrive at a state where every subset A of U of size 51 or more has $f(A) = 0$, and thus $F = \sum_{|T|=50} f(T)$. But the number of integers that are in every set in T is not zero, so $f(T) > 0$ whenever $|T| = 50$, so by the divisibility requirement, we have $f(T) \geq 50$. This yields the lower bound

$$F \geq \sum_{|T|=50} 50 = 50\binom{100}{50}.$$

We now provide a construction that achieves this bound. For that purpose, we introduce the sequence $(m_k)_{k=0}^{100}$ by setting $m_0 = 100$ and defining m_k recursively as the smallest nonnegative integer for which $m_k + \sum_{i=1}^{k} \binom{k}{i} m_{k-i}$ is divisible by $100 - k$. Then, for each subset I of $\{1, 2, \dots, 100\}$ of size $100 - k$, we choose m_k integers so that all the integers chosen are pairwise distinct, and place these integers in S_i whenever $i \in I$. This is a valid construction as the intersection of the 100 sets is not empty, and the divisibility requirement is satisfied since for any subset T of $\{S_1, S_2, \dots, S_{100}\}$ of size $100 - k$, we have

$$F(T) = \sum_{T \subseteq A} f(A) = f(T) + \sum_{T \subsetneq A} f(A) = m_k + \sum_{i=1}^{k} \binom{k}{i} m_{k-i},$$

which is divisible by $|T|$ by definition.

Next, we prove that $m_k = 100 - 2k$ for $k = 0, 1, \ldots, 50$. Indeed,

$$\begin{aligned}\sum_{i=0}^{k} \binom{k}{i}(100-2k+2i) &= (100-2k)\sum_{i=0}^{k}\binom{k}{i} + 2\sum_{i=0}^{k} i\binom{k}{i} \\ &= (100-2k)\sum_{i=0}^{k}\binom{k}{i} + 2k\sum_{i=1}^{k}\binom{k-1}{i-1} \\ &= (100-2k)\cdot 2^k + 2k\cdot 2^{k-1} \\ &= (100-k)\cdot 2^k,\end{aligned}$$

which is divisible by $100 - k$; since $100 - 2k < 100 - k$ for all $1 \le k \le 50$, we must then have $m_k = 100 - 2k$, as claimed.

We can readily verify that that our construction yields the lower bound for F:

$$\begin{aligned}\sum_{|T|\ge 50} f(T) &= \sum_{k=0}^{50}\binom{100}{100-k}(100-2k) \\ &= 100\sum_{k=0}^{50}\binom{100}{100-k} - 2\sum_{k=0}^{50}\binom{100}{100-k}k \\ &= 100\sum_{k=0}^{50}\binom{100}{k} - 200\sum_{k=0}^{49}\binom{99}{k} \\ &= 100\cdot\frac{2^{100}+\binom{100}{50}}{2} - 200\cdot\frac{2^{99}}{2} \\ &= 50\binom{100}{50}.\end{aligned}$$

Our proof is now complete.

Note. It may be interesting to see all terms of the sequence $(m_k)_{k=0}^{100}$ explicitly: 100, 98, 96, 94, 92, 90, 88, 86, 84, 82, 80, 78, 76, 74, 72, 70, 68, 66, 64, 62, 60, 58, 56, 54, 52, 50, 48, 46, 44, 42, 40, 38, 36, 34, 32, 30, 28, 26, 24, 22, 20, 18, 16, 14, 12, 10, 8, 6, 4, 2, 0, 47, 40, 29, 14, 40, 4, 3, 18, 22, 20, 22, 4, 5, 4, 15, 16, 17, 28, 6, 0, 14, 0, 18, 22, 0, 12, 4, 18, 9, 0, 2, 16, 15, 12, 0, 2, 1, 8, 3, 0, 0, 4, 6, 0, 0, 0, 0, 0, 0. While we have a simple formula for m_k when $0 \le k \le 50$, the terms appear quite chaotic after that.

Problem 3 $(212; 8, 1, 4, 4, 2, 5, 4, 184)$; *proposed by Krit Boonsiriseth*. Let (m, n) be positive integers with $n \ge 3$ and draw a regular n-gon. We wish to triangulate this n-gon into $n - 2$ triangles, each colored one of m colors, so that each color has an equal sum of areas. For which (m, n) is such a triangulation and coloring possible?

Solution. We claim that such a triangulation and coloring is possible if and only if m is a proper divisor of n.

We will be working in the complex plane; in particular, we will use the complex number

$$\omega = \cos(2\pi/n) + i\sin(2\pi/n),$$

which is a primitive n^{th} root of unity. Then when the n complex numbers

$$1,\ \omega,\ \ldots,\ \omega^{n-1}$$

are drawn as points in the complex plane, they form the vertices of a regular n-gon inscribed in the unit circle centered at the origin. We will take these as the vertices of our regular n-gon and do all our calculations using this as a reference.

Before we start, let us recall how to calculate areas in the complex plane. We adopt the convention that areas are *signed*: meaning that the area of a triangle $P_1P_2P_3$ is considered positive if P_1, P_2, and P_3 are labeled in counterclockwise order, and negative otherwise. The formula (see [**2**, Theorem 6.7]) for the signed area of the triangle whose vertices are given by three complex numbers z_1, z_2, and z_3 in the complex plane is given by the 3×3 determinant

$$\text{Area}(z_1, z_2, z_3) = \frac{i}{4} \det \begin{bmatrix} z_1 & \bar{z_1} & 1 \\ z_2 & \bar{z_2} & 1 \\ z_3 & \bar{z_3} & 1 \end{bmatrix},$$

where $\bar{z}$ is the complex conjugate of z. (We allow degenerate triangles with zero area.) In particular, for the triangle with vertices $\omega^a, \omega^b, \omega^c$ (where $a, b, c \in \mathbb{N}_0$), we have

$$\begin{aligned} \text{Area}(\omega^a, \omega^b, \omega^c) &= \frac{i}{4} \det \begin{bmatrix} \omega^a & \omega^{-a} & 1 \\ \omega^b & \omega^{-b} & 1 \\ \omega^c & \omega^{-c} & 1 \end{bmatrix} \\ &= \frac{i}{4} \left(\omega^{a-b} + \omega^{b-c} + \omega^{c-a} - \omega^{a-c} - \omega^{b-a} - \omega^{c-b} \right). \end{aligned}$$

We now provide an example triangulation and coloring when m is a proper divisor of n. The construction is easy to describe: it suffices to actually just take all the diagonals from one particular vertex, say 1, and then color the triangles with the m colors in cyclic order. For example, when $n = 9$ and $m = 3$, a coloring with red, green, blue would yield Figure 1.

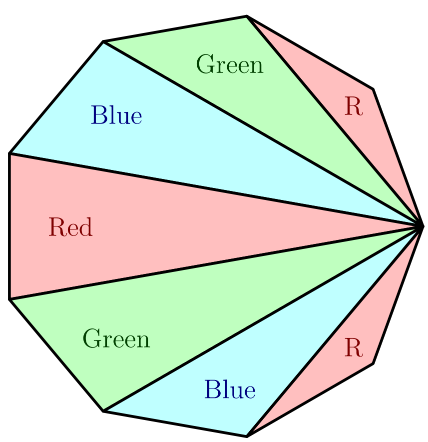

Figure 1 Illustration for Problem 3.

To verify that this construction has the desired property, let us fix a residue $r \bmod m$ corresponding to one of the colors. Then the sum of the areas for this color equals

$$\sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \operatorname{Area}(\omega^0,\omega^j,\omega^{j+1})$$
$$= \frac{i}{4} \sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \left(\omega^{-j}+\omega^{-1}+\omega^{j+1}-\omega^{-j-1}-\omega^{j}-\omega\right).$$

However, if m is a proper divisor of n, then

$$\sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \omega^j = \omega^r(1+\omega^m+\omega^{2m}+\cdots+\omega^{n-m}) = 0,$$

since the sum of $\frac{n}{m}$ equally spaced complex numbers around the unit circle equals zero. For the same reason, we have

$$\sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \omega^{-j} = \sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \omega^{j+1} = \sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \omega^{-j-1} = 0.$$

Therefore, we get

$$\sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \operatorname{Area}(\omega^0,\omega^j,\omega^{j+1}) = \frac{i}{4} \sum_{\substack{0\le j<m\\ j\equiv r \bmod m}} \left(\omega^{-1}-\omega\right)$$
$$= \frac{i}{4}\cdot\frac{n}{m}\left(\omega^{-1}-\omega\right) = \frac{n}{2m}\sin\left(\frac{2\pi}{n}\right).$$

Because this value does not depend on the residue r, all colors have equal area.

We now prove that n being divisible by m is necessary. This is certainly true for $n=3$ and $n=4$ since we must have $m\le n-2$. For the same reason, for $n=6$ we must have $m\le 4$, and we can rule out $(n,m)=(6,4)$ by pointing out that in a triangulation of a regular hexagon, some (at least two) triangles are formed by three consecutive vertices, and the area of such a triangle is only $1/6$ of the area of the hexagon. Therefore, below we assume that $n\notin\{3,4,6\}$.

Before our proof, we revisit some topics from algebraic number theory.[2] Recall that a complex number is *algebraic* if it is a root of a nonzero polynomial with rational number coefficients, and an *algebraic integer* is a complex number that is a root of a monic polynomial with integer coefficients. For example, every rational number r is algebraic as it is the root of the polynomial $x-r$, but it is not an algebraic integer unless it is an integer; i is an algebraic integer as it is a root of x^2+1, and ω introduced above is an algebraic integer since it is a root of x^n-1. An important fact is that the collection of algebraic integers forms a ring, which we denote here by $\overline{\mathbb{Z}}$; in particular, the sums and products of algebraic integers are also algebraic integers.

Now let P_ω be the *minimal polynomial* of ω, that is, the lowest-degree nonzero polynomial with rational coefficients and ω as a root. The degree d of P_ω then is given by $d=\varphi(n)$ where φ is the Euler totient function. As is well known, $\varphi(n)\ge 3$ when $n=5$ or $n\ge 7$.

Let $\mathbb{Q}(\omega)$ be the set (indeed, the *field*) of complex numbers that can be written as polynomials in ω with rational number coefficients and of degree at most $d-1$:

$$\mathbb{Q}(\omega) = \left\{a_0+a_1\omega+\cdots+a_{d-1}\omega^{d-1} \mid a_0,\ldots,a_{d-1}\in\mathbb{Q}\right\}.$$

[2]For these and other properties of algebraic numbers and algebraic integers, see [**4**], for example.

It is an important fact that these representations are unique, and the set of algebraic integers inside $\mathbb{Q}(\omega)$ is exactly the ones with integer coefficients:

$$\overline{\mathbb{Z}} \cap \mathbb{Q}(\omega) = \left\{a_0 + a_1\omega + \cdots + a_{d-1}\omega^{d-1} \mid a_0, \ldots, a_{d-1} \in \mathbb{Z}\right\}.$$

Suppose now that a valid triangulation and coloring of the regular n-gon exists. (We still refer to the n-gon whose vertices are ω^k with $k = 0, 1, \ldots, n-1$.) As we have shown above, the area of each of the $n-2$ triangles in the triangulation is of the form

$$\text{Area}(\omega^a, \omega^b, \omega^c) = \frac{i}{4}\left(\omega^{a-b} + \omega^{b-c} + \omega^{c-a} - \omega^{a-c} - \omega^{b-a} - \omega^{c-b}\right)$$

for some integers a, b, and c, and thus $\frac{i}{4}$ times an algebraic integer. Therefore, for each color, the sum of the areas of the triangles of that color is also $\frac{i}{4}$ times an algebraic integer. Since that sum must equal

$$\frac{i}{4} \cdot \frac{n}{m}\left(\omega^{-1} - \omega\right) = \frac{n}{2m}\sin\left(\frac{2\pi}{n}\right),$$

the number $\frac{n}{m}\left(\omega^{-1} - \omega\right)$ must be an algebraic integer, and hence $\frac{n}{m}\omega^2 - \frac{n}{m}$ is an algebraic integer as well. However, if

$$\frac{n}{m}\omega^2 - \frac{n}{m} \in \overline{\mathbb{Z}} \cap \mathbb{Q}(\omega),$$

then n/m is an integer, which proves our claim.

Problem 4 $(268; 157, 22, 21, 10, 4, 28, 9, 17)$; *proposed by Rishabh Das*. Let m and n be positive integers. A circular necklace contains mn beads, each either red or blue. It turned out that no matter how the necklace was cut into m blocks of n consecutive beads, each block had a distinct number of red beads. Determine, with proof, all possible values of the ordered pair (m, n).

Solution. The number of red beads among n beads (consecutive or not) is a number between 0 and n, inclusive. Therefore, by the Pigeonhole Principle, if $m > n+1$, then at least two of the blocks of n consecutive beads must have the same number of red beads. This implies that $m \leq n+1$. We will now prove that whenever $m \leq n+1$, there exists a necklace made of mn beads with the required property.

To describe our coloring, we take a necklace containing mn beads. Starting with an arbitrary bead and moving clockwise, we color the first n beads blue, followed by $n-1$ blue beads and then a red bead, then $n-2$ blue beads followed by two red beads, and so on, ending with $n-m+1$ blue beads followed by $m-1$ red beads. We can visualize this construction in reading order of an array of m rows and n columns, as illustrated below for $n = 8$ and $m = 6$.

B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	R
B	B	B	B	B	B	R	R
B	B	B	B	B	R	R	R
B	B	B	B	R	R	R	R
B	B	B	R	R	R	R	R

Cutting the necklace into m blocks of n consecutive beads will then result in blocks $T_1, T_2, \ldots, T_m$, where for a given value of $t = 1, 2, \ldots, n$, the block T_i consists of the last t beads in row i followed by the first $n - t$ beads in row $i + 1$ (with $i = 1, 2, \ldots, m$ considered mod m).

We can count the number of red beads in the blocks, as follows. Let $1 \le i \le m$.

- If T_i starts with a blue bead, it must include all red beads in row i but no other red beads. Therefore, it has exactly $i - 1$ red beads.
- If T_i starts with a red bead for some $i \le m - 1$, then it also ends in a red bead, so it contains the blue beads in row $(i + 1)$ but no other blue beads. Therefore, it has $n - i$ blue beads and thus i red beads.
- If T_m starts with a red bead, then it includes all red beads in row m but no other red beads. Therefore, it has exactly t red beads.

To establish that our construction works, we consider three cases. If $t \ge m$, then each block starts with a blue bead, so the number of red beads in the blocks are $0, 1, \ldots, m - 1$, respectively. If $t = m - 1$, then each block except for T_m starts with a blue bead while T_m starts with a red bead, and thus the number of red beads in the blocks are again $0, 1, \ldots, m - 1$, respectively. Finally, in the case when $t \le m - 2$, blocks $T_1, T_2, \ldots, T_t$ start with a blue bead while $T_{t+1}, T_{t+2}, \ldots, T_m$ start with a red bead; therefore, the number of red beads in the blocks are, in order,

$$0, 1, \ldots, t - 1, t + 1, t + 2, \ldots, m - 1, t.$$

This proves that in each case the blocks have a pairwise distinct number of red beads, as claimed.

Problem 5 $(237; 28, 0, 0, 0, 0, 2, 3, 204)$; *proposed by Anton Trygub*. Point D is selected inside acute triangle ABC so that $\angle DAC = \angle ACB$ and $\angle BDC = 90^\circ + \angle BAC$. Point E is chosen on ray BD so that $AE = EC$. Let M be the midpoint of BC. Show that line AB is tangent to the circumcircle of triangle BEM.

Solution. As $EA = EC$, we know E lies on the perpendicular bisector of AC. We reflect point B across this perpendicular bisector to a point Q, and we reflect B across E to a point F. Note that this gives an isosceles (and hence cyclic) trapezoid $ABQC$, and that lines QF and AC are perpendicular.

We deal with the configuration shown in Figure 2, where we assume $AB < BC < AC$, so that B, A, D lie on one side of the perpendicular bisector of AC, and F, Q, M, and C lie on the other side. The proofs of the remaining cases are handled analogously (alternatively, experts may prefer to take directed angles modulo 180° to avoid the case distinction).

The condition that $\angle DAC = \angle ACB$ implies that D lies on ray AQ, and the condition that $\angle BDC = 90^\circ + \angle BAC$ implies that

$$\angle FDC = 180^\circ - \angle BDC = 90^\circ - \angle BAC = 90^\circ - \angle QCA = \angle FQC.$$

Therefore, the four points F, D, Q, C lie on a circle ω, and thus $\angle DFC$ and $\angle DQC$ are supplementary angles.

Since E and M are the midpoints of BF and BC, respectively, we also know that lines EM and FC are parallel. Therefore,

$$\angle BEM = \angle BFC = \angle DFC = 180^\circ - \angle DQC.$$

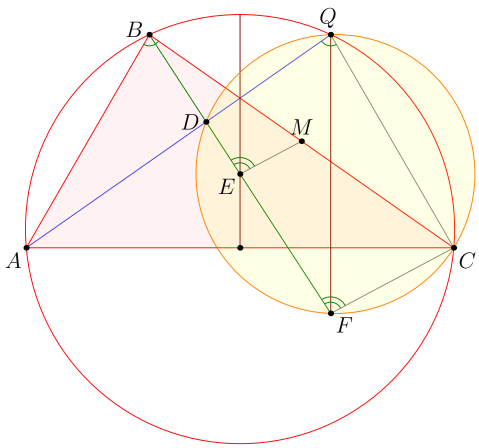

Figure 2 Illustration for Problem 5.

But

$$\angle DQC = \angle AQC = \angle ABC = \angle ABM,$$

so $\angle BEM = 180^\circ - \angle ABM$.

To complete our proof, consider the tangent line ℓ to the circumcircle of triangle BEM at B. Note that the angle between ℓ and BM equals the inscribed angle $\angle BEM$ in the circle. This means that ℓ must coincide with line AB, as claimed.

Problem 6 $(126; 4, 1, 0, 0, 0, 8, 12, 101)$; *proposed by Titu Andreescu and Gabriel Dospinescu*. Let $n > 2$ be an integer and let $\ell \in \{1, 2, \ldots, n\}$. A collection $A_1, \ldots, A_k$ of (not necessarily distinct) subsets of $\{1, 2, \ldots, n\}$ is called *ℓ-large* if $|A_i| \geq \ell$ for all $1 \leq i \leq k$. Find, in terms of n and ℓ, the largest real number c such that the inequality

$$\sum_{i=1}^{k} \sum_{j=1}^{k} x_i x_j \frac{|A_i \cap A_j|^2}{|A_i| \cdot |A_j|} \geq c \left(\sum_{i=1}^{k} x_i \right)^2$$

holds for all positive integers k, all nonnegative real numbers $x_1, \ldots, x_k$, and all ℓ-large collections $A_1, \ldots, A_k$ of subsets of $\{1, 2, \ldots, n\}$.

Note: For a finite set S, $|S|$ denotes the number of elements in S.

Solution. Letting $a_i = x_i / |A_i|$, the inequality becomes

$$\sum_{i,j=1}^{k} a_i a_j |A_i \cap A_j|^2 \geq c \left(\sum_{i=1}^{k} a_i |A_i| \right)^2 . \tag{1}$$

We first prove that with

$$c = \frac{1}{n} + \frac{(\ell - 1)^2}{n(n-1)},$$

(1) holds for all positive integers k, all nonnegative real numbers $a_1, \ldots, a_k$, and all ℓ-large collections $A_1, \ldots, A_k$ of subsets of $\{1, 2, \ldots, n\}$.

It will be convenient to use indicator functions. For an element $m \in \{1, 2, \ldots, n\}$, we let $\mathbb{1}_{A_i}(m) = 1$ when $m \in A_i$ and $\mathbb{1}_{A_i}(m) = 0$ when $m \notin A_i$.

Then the left side of (1) can be rewritten as

$$\begin{aligned}\sum_{i,j=1}^{k} a_i a_j |A_i \cap A_j|^2 &= \sum_{i,j=1}^{k} a_i a_j \left(\sum_{m=1}^{n} \mathbb{1}_{A_i}(m) \cdot \mathbb{1}_{A_j}(m)\right)^2 \\ &= \sum_{i,j=1}^{k} a_i a_j \sum_{m_1,m_2=1}^{n} \mathbb{1}_{A_i}(m_1) \cdot \mathbb{1}_{A_i}(m_2) \cdot \mathbb{1}_{A_j}(m_1) \cdot \mathbb{1}_{A_j}(m_2) \\ &= \sum_{m_1,m_2=1}^{n} \sum_{i,j=1}^{k} \left(a_i \cdot \mathbb{1}_{A_i}(m_1) \cdot \mathbb{1}_{A_i}(m_2)\right) \cdot \left(a_j \cdot \mathbb{1}_{A_j}(m_1) \cdot \mathbb{1}_{A_j}(m_2)\right) \\ &= \sum_{m_1,m_2=1}^{n} \left(\sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m_1) \cdot \mathbb{1}_{A_i}(m_2)\right)^2 \\ &= \sum_{m=1}^{n} \left(\sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m)\right)^2 + \sum_{\substack{1 \le m_1, m_2 \le n \\ m_1 \ne m_2}} \left(\sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m_1) \cdot \mathbb{1}_{A_i}(m_2)\right)^2 .\end{aligned} \tag{2}$$

Applying the Quadratic Mean–Arithmetic Mean Inequality then yields

$$\sum_{m=1}^{n} \left(\sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m)\right)^2 \ge \frac{1}{n} \left(\sum_{m=1}^{n} \sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m)\right)^2 , \tag{3}$$

and

$$\begin{aligned}&\sum_{\substack{1 \le m_1, m_2 \le n \\ m_1 \ne m_2}} \left(\sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m_1) \cdot \mathbb{1}_{A_i}(m_2)\right)^2 \\ &\qquad \ge \frac{1}{n(n-1)} \left(\sum_{\substack{1 \le m_1, m_2 \le n \\ m_1 \ne m_2}} \sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m_1) \cdot \mathbb{1}_{A_i}(m_2)\right)^2 .\end{aligned} \tag{4}$$

But

$$\sum_{m=1}^{n} \sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m) = \sum_{i=1}^{k} a_i \sum_{m=1}^{n} \mathbb{1}_{A_i}(m) = \sum_{i=1}^{k} a_i |A_i|,$$

so (3) becomes

$$\sum_{m=1}^{n} \left(\sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m)\right)^2 \ge \frac{1}{n} \left(\sum_{i=1}^{k} a_i |A_i|\right)^2 . \tag{5}$$

Also,

$$\sum_{\substack{1\le m_1,m_2\le n\\ m_1\ne m_2}} \sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m_1)\cdot \mathbb{1}_{A_i}(m_2)$$
$$= \sum_{i=1}^{k} a_i \sum_{\substack{1\le m_1,m_2\le n\\ m_1\ne m_2}} \mathbb{1}_{A_i}(m_1)\cdot \mathbb{1}_{A_i}(m_2) = \sum_{i=1}^{k} a_i|A_i|(|A_i|-1),$$

and since $|A_i|-1\ge \ell-1$ for all i, from (4), we get

$$\sum_{\substack{1\le m_1,m_2\le n\\ m_1\ne m_2}} \left(\sum_{i=1}^{k} a_i \cdot \mathbb{1}_{A_i}(m_1)\cdot \mathbb{1}_{A_i}(m_2)\right)^2 \ge \frac{(\ell-1)^2}{n(n-1)}\left(\sum_{i=1}^{k} a_i|A_i|\right)^2. \quad (6)$$

Combining (5) and (6) in (2), we obtain

$$\sum_{i,j=1}^{k} a_i a_j |A_i\cap A_j|^2 \ge \left(\frac{1}{n}+\frac{(\ell-1)^2}{n(n-1)}\right)\left(\sum_{i=1}^{k} a_i|A_i|\right)^2,$$

as claimed.

To prove that the value of c cannot be lowered, we exhibit an instance where equality holds in (1) with

$$c=\frac{1}{n}+\frac{(\ell-1)^2}{n(n-1)}.$$

Namely, we let $k=\binom{n}{\ell}$, $\{A_i \mid i=1,2,\dots,k\}$ be the collection of subsets of $\{1,2,\dots,n\}$ with exactly ℓ elements, and $a_i=1$ for all i, and prove that

$$\sum_{i,j=1}^{k} |A_i\cap A_j|^2 = \left(\frac{1}{n}+\frac{(\ell-1)^2}{n(n-1)}\right)k^2\ell^2. \quad (7)$$

Observe first that, by symmetry, $\sum_{i=1}^{k}|A_i\cap A_j|^2$ is constant for any $1\le j\le k$, so

$$\sum_{i,j=1}^{k} |A_i\cap A_j|^2 = k\cdot\sum_{i=1}^{k}|A_i\cap\{1,2,\dots,\ell\}|^2.$$

Therefore, to prove (7), we need to establish that

$$\sum_{i=1}^{k}|A_i\cap\{1,2,\dots,\ell\}|^2 = \left(\frac{1}{n}+\frac{(\ell-1)^2}{n(n-1)}\right)k\ell^2. \quad (8)$$

Consider the $k\times\ell$ matrix where the entry $t_{i,j}$ in the ith row and jth column is 1 if $j\in A_i$ and 0 otherwise. We then have

$$\sum_{i=1}^{k}|A_i\cap\{1,2,\dots,\ell\}|^2 = \sum_{i=1}^{k}\left(\sum_{j=1}^{\ell} t_{i,j}\right)^2 = \sum_{i=1}^{k}\sum_{j=1}^{\ell} t_{i,j}^2 + \sum_{i=1}^{k}\sum_{\substack{1\le j_1,j_2\le\ell\\ j_1\ne j_2}} t_{i,j_1}t_{i,j_2}.$$

Now

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} t_{i,j}^2 = \sum_{j=1}^{\ell}\sum_{i=1}^{k} t_{i,j}^2 = \sum_{j=1}^{\ell}\sum_{i=1}^{k} t_{i,j} = \binom{n-1}{\ell-1}\ell,$$

since each value of j appears in $\binom{n-1}{\ell-1}$ sets A_i; similarly,

$$\sum_{i=1}^{k}\sum_{\substack{1\le j_1,j_2\le \ell\\ j_1\ne j_2}} t_{i,j_1}t_{i,j_2} = \sum_{\substack{1\le j_1,j_2\le \ell\\ j_1\ne j_2}}\sum_{i=1}^{k} t_{i,j_1}t_{i,j_2} = \binom{n-2}{\ell-2}\ell(\ell-1).$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{k} |A_i \cap \{1,2,\ldots,\ell\}|^2 &= \binom{n-1}{\ell-1}\ell + \binom{n-2}{\ell-2}\ell(\ell-1)\\ &= \left(\frac{1}{n} + \frac{(\ell-1)^2}{n(n-1)}\right)\binom{n}{\ell}\ell^2,\end{aligned}$$

as claimed in (8). This completes our proof.

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REFERENCES

1. Joseph Bertrand, *Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu'elle renferme.*, Journal de l'ecole imperiale politechnique **18** (1845), 123–140 (French).
2. Evan Chen, *Euclidean geometry in mathematical olympiads*, AMS/MAA Problem Books Series, Mathematical Association of America, Washington, DC, 2016.
3. Pál Erdős, *Proof of a theorem of Chebyshev.*, Acta Litt. Sci. Szeged **5** (1932), 194–198 (German).
4. Róbert Freud and Edit Gyarmati, *Number theory*, revised ed., Pure and Applied Undergraduate Texts, vol. 48, American Mathematical Society, Providence, RI, [2020] ©2020.
5. Tchebichef, *Mémoire sur les nombres premiers*, Journal de Mathématiques Pures et Appliquées **1e série, 17** (1852), 366–390 (French).

Summary. We present the problems and solutions to the 53rd Annual United States of America Mathematical Olympiad.

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