Playing with Triangular Numbers

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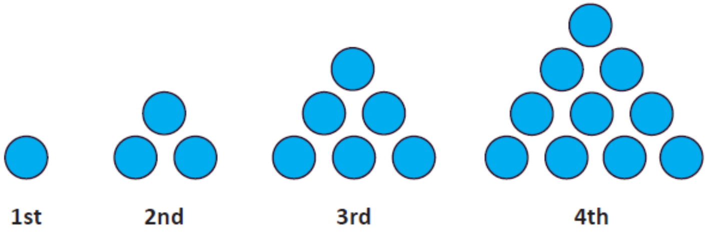
Triangular Numbers

What are triangular numbers?



Triangular Numbers

The n-th triangular number, Δ_n is $\dfrac{n(n+1)}{2}$

Classic proof:

$$\begin{aligned}
\Delta_n &= 1 + 2 + \cdots + n \\
\Delta_n &= n + (n-1) + \cdots + 1 \\
2\Delta_n &= (n+1) + (n+1) + \cdots + (n+1) \\
2\Delta_n &= n(n+1) \\
\Delta_n &= \frac{n(n+1)}{2}.
\end{aligned}$$

Combinatorial Proof:

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \sum_{j=0}^{i-1} 1 = \binom{n+1}{2}.$$

Probabilistic Proof

Let X be the sum of two uniformly-distributed n-sided dice.

$$\mathbb{P}[X=k] = \begin{cases} \frac{(k-1)}{n^2}, & 2 \le k \le n+1 \\ \frac{n-i+1}{n^2}, & k = n+i \text{ with } 2 \le i \le n \end{cases}$$

Since $2 \le X \le 2n$, then

$$1 = \sum_{k=2}^{2n} \mathbb{P}[X=k] = \sum_{k=2}^{n+1} \frac{k-1}{n^2} + \sum_{i=2}^{n} \frac{n-i+1}{n^2}$$

$$1 = \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}\right) + \left(\frac{n-1}{n^2} + \frac{n-2}{n^2} + \cdots + \frac{1}{n^2}\right)$$

$$n^2 = (1+2+\cdots+n) + (1+2+\cdots+(n-1))$$

$$n^2 = 2(1+2+\cdots+n) - n.$$

Playing with Triangular Numbers

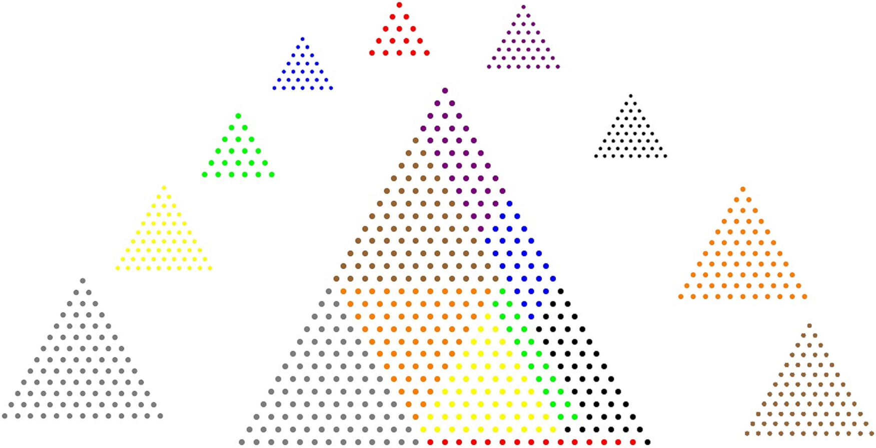
$$1 + 3 + 6 = 10$$

McMullen, inspired by this, asked himself:

- For which k can we find k consecutive triangular numbers that add up to be a triangular number?
- Can we find the solutions?

McMullen showed there are infinitely many solutions for $k = 2, 3, 5$, but no solutions for $k = 4$.

Example



$k = 4$ case

Elementary manipulations show that the sum of the k-consecutive triangular numbers starting at Δ_n is Δ_m whenever

$$(2m+1)^2 - k(2n+k)^2 = \frac{(k-1)(k^2+k-3)}{3}.$$

When $k = 4$ we get

$$(2m+1-4n-8)(2m+1+4n+8) = 17.$$

From which

$$(m,n) = (4,0), (4,-4), (-5,0), \text{ and } (-5,-4).$$

Square k

Theorem

Let $k > 4$ be a square. Then there exist k consecutive triangular numbers that add up to make a bigger triangular number.

Useful Factorization

Let $k = a^2$.

Then

$$(2m+1)^2 - a^2(2n+a^2)^2 = \frac{a^6 - 4a^2 + 3}{3}.$$

$$(2m+1+2na+a^3)(2m+1+2na+a^3) = \frac{(a+1)(a-1)\left(a^4+a^2-3\right)}{3}$$

k an even square

$$(2m+1-2na-a^3)(2m+1+2na+a^3) = \frac{a^6-4a^2+3}{3}$$

Solving

$$2m+1-2na-a^3 = 1$$
$$2m+1+2na+a^3 = \frac{a^6-4a^2+3}{3}$$

yields

$$m = \frac{(a^4-4)(a^2)}{12}$$
$$n = \frac{a(a^4-6a-4)}{12}$$

k an odd square

We'll consider three cases:

$$a \equiv 0 \pmod 3$$

$$a \equiv 1 \pmod 3$$

$$a \equiv 2 \pmod 3$$

$a \equiv 1 \bmod 3$ or $a \equiv 0 \bmod 3$

$$(2m+1+2na+a^3)(2m+1+2na+a^3) = \frac{(a+1)\,(a-1)\,\left(a^4+a^2-3\right)}{3}$$

Solving

$$2m+1-2na-a^3 = a+1$$
$$2m+1+2na+a^3 = \frac{(a-1)(a^4+a^2-3)}{3}$$

yields

$$m = \frac{a^2(a-1)(a^2+1)}{12}$$
$$n = \frac{(a+2)(a-3)(a^2+1)}{12}$$

$a \equiv 2 \bmod 3$ or $a \equiv 0 \bmod 3$

$$(2m+1+2na+a^3)(2m+1+2na+a^3) = \frac{(a+1)(a-1)\left(a^4+a^2-3\right)}{3}$$

Solving

$$2m+1-2na-a^3 = a-1$$
$$2m+1+2na+a^3 = \frac{(a+1)(a^4+a^2-3)}{3}$$

yields

$$m = \frac{a^5+a^4+a^3+a^2-12}{12}$$
$$n = \frac{(a+3)(a-2)(a^2+1)}{12}$$

$k = 6$

Recall
$$(2m+1)^2 - k(2n+k)^2 = \frac{(k-1)(k^2+k-3)}{3}.$$
When $k = 6$:
$$x^2 - 6y^2 = 65.$$
Therefore $x^2 \equiv 6y^2 \bmod 13$. But
$$\left(\frac{6}{13}\right) = -1.$$
Therefore, there are no solutions for $k \equiv 6 \bmod 13$.

A sufficient condition for k

Lemma

Let $q > 3$ be a prime number. Suppose that $k \in \mathbb{Z}$ is such that

1. *k is not a square modulo q,*
2. *$q \parallel k^2 + k - 3$.*

Then there are no k consecutive triangular numbers that add up to a triangular number.

Example:

If $k \equiv 45 \mod 53$ and $k \not\equiv 2430 \mod 53^2$. There are 52 residues modulo 53^2 which satisfy these conditions.

Main Theorem

Theorem

Let $K(x)$ be the number of k's such that $k \leq x$ that have solutions. Then:

$$\sqrt{x} \leq K(x) \ll \frac{x}{\sqrt{\log(x)}}.$$

Note: $f(x) \ll g(x)$ if there exists a positive constant C such that $f(x) \leq C|g(x)|$ for large enough x.

Remark

This theorem implies that the existence of k consecutive triangular numbers that add up to a triangular number is a rare occurrence.

Finding q such that $q \parallel k^2+k-3$ and $(k/q)=-1$

If $q \neq 13$, $k^2+k-3 \equiv 0 \mod q$ has two distinct solutions k_1, k_2 whenever $(13/q)=1$. We then have three possibilities

1. Both k_1, k_2 are squares modulo q.
2. One of k_1, k_2 is a square and the other one isn't.
3. Neither k_1, k_2 are squares modulo q.

A pair of important sets of primes

Let $\mathcal{A}$ be the set of primes q for which we have k_1, k_2 both nonsquares modulo q.

Let $\mathcal{B}$ be the set of primes q for which exactly one of k_1, k_2 is a square modulo q.

If $q \in \mathcal{A}$, then the proportion of residues modulo q^2 one must avoid are

$$2\frac{q-1}{q^2} = \frac{2}{q} - \frac{2}{q^2}.$$

If $q \in \mathcal{B}$, then the proportion of residues modulo q^2 one must avoid are

$$\frac{q-1}{q^2} = \frac{1}{q} - \frac{1}{q^2}.$$

A useful example

Consider the polynomial $x^2 + 1$.

- $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$
- $x^2 + 1 = (x + 1)^2$ in $\mathbb{Z}_2[x]$
- $x^2 + 1 = (x + 2)(x + 3)$ in $\mathbb{Z}_5[x]$
- $x^2 + 1 = (x + 5)(x + 8)$ in $\mathbb{Z}_{13}[x]$
- $x^2 + 1$ can be factored in $\mathbb{Z}_p[x]$ whenever $p \equiv 1 \bmod 4$. In fact

$$x^2 + 1 = \left(x - \left(\frac{p-1}{2}\right)!\right)\left(x + \left(\frac{p-1}{2}\right)!\right) \bmod p$$

- $x^2 + 1$ is irreducible in $\mathbb{Z}_p[x]$ for $p \equiv 3 \bmod 4$.

$$\alpha^2 \equiv -1 \bmod p \Leftrightarrow \alpha^{p-1} \equiv (-1)^{(p-1)/2} \bmod p$$
$$\Leftrightarrow (-1)^{(p-1)/2} \equiv 1 \bmod p \Leftrightarrow p \equiv 3 \bmod 4.$$

Quantifying the proportion of primes in $\mathcal{A}$, $\mathcal{B}$

Consider $f(x) = x^4 + x^2 - 3$. Let's analyze how $f(x)$ might factor in $\mathbb{Z}_q$. There are several possibilities

- (1,1,1,1)
- (1,1,2)
- (2,2)
- 4

Primes in $\mathcal{B}$ would split as (1,1,2).

Primes in $\mathcal{A}$ would be primes that are squares modulo 13 and that don't split as (1,1,2) or (1,1,1,1).

Chebotarev Density Theorem

Theorem

Suppose that $f(x) \in \mathbb{Z}[x]$ is monic and irreducible over $\mathbb{Q}$, with $\deg f(x) = n$. Let $\mathbb{L}$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Fix a partition $\langle k_1, \ldots, k_r \rangle$ of n (that is, a tuple of positive integers $k_1 \geq k_2 \geq \cdots \geq k_r$ with $k_1 + \cdots + k_r = n$). Let δ be the proportion of elements of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ which, when viewed as permutations on the roots of $f(x)$, have cycle type $\langle k_1, \ldots, k_r \rangle$. For all but finitely many primes p, the polynomial $f(x)$ factors as a product of distinct monic irreducible polynomials modulo p, and δ is the proportion of primes for which these irreducibles have degrees $k_1, \ldots, k_r$.

Chebotarev in our problem

Consider $f(x) = x^4 + x^2 - 3$. f is irreducible over $\mathbb{Q}$, let $\mathbb{L}$ be the splitting field of f over $\mathbb{Q}$, then $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ is isomorphic to

$$\{(1), (1324), (12)(34), (1423), (34), (13)(24), (12), (14)(23)\}.$$

- 1 of the 8 elements decomposes as (1,1,1,1)
- 3 of the 8 elements decompose as (2,2)
- 2 of the 8 elements decompose as (1,1,2)
- 2 of the 8 elements decompose as (4)

The proportion of primes $q \in \mathcal{B}$ (one root a square, the other root a non-square) is $2/8 = 1/4$.

The proportion of primes $q \in \mathcal{A}$ (both roots non-squares) is $1/2 - 1/8 - 2/8 = 1/8$.

Idea of Proof

There are several residues modulo certain squares of primes that must be avoided for k to be able to yield solutions.
We then get the following upper bound heuristic:

$$
\begin{aligned}
K(x) &\ll x \prod_{\substack{q \leq x \\ q \in \mathcal{A}}} \left(1 - \frac{2}{q} + \frac{2}{q^2}\right) \prod_{\substack{q \leq x \\ q \in \mathcal{B}}} \left(1 - \frac{1}{q} + \frac{1}{q^2}\right) \\
&\ll x \left(\frac{1}{(\log^{1/8} x)^2}\right)\left(\frac{1}{\log^{1/4}(x)}\right) \\
&\ll \frac{x}{\sqrt{\log x}}.
\end{aligned}
$$

Non-Chebotarev (weaker) proof

One could avoid the use of Chebotarev to get that $K(x) = o(x)$. Namely, the primes in $\mathcal{B}$ can be characterized as primes q satisfying

$$\left(\frac{13}{q}\right) = 1 \quad \& \quad \left(\frac{-3}{q}\right) = -1.$$

The primes in $\mathcal{B}$ have proportion 1/4.

Then

$$K(x) \ll \frac{x}{(\log(x))^{1/4}}.$$

Main Theorem

Theorem

Let $K(x)$ be the number of k's such that $k \leq x$ that have solutions. Then:

$$\sqrt{x} \leq K(x) \ll \frac{x}{\sqrt{\log(x)}}.$$

Corollary

$K(x) = o(x)$, i.e., the natural density of k's for which there is a solution is 0.

On the hunt for a lower bound

Our goal is solving

$$(2m+1)^2 - k(2n+k)^2 = \frac{(k-1)(k^2+k-3)}{3}.$$

Let p be a prime number satisfying:

1. $p \equiv 7 \bmod 24$
2. $p^2 + p - 3$ is not divisible by any prime q for which $p \bmod q$ is a nonsquare
3. $\mathbb{Q}(\sqrt{p})$ has class number 1.

Then there exist p consecutive triangular numbers that add up to a triangular number.

$k = 127$

We want to solve

$$(2m+1)^2 - 127(2n+127)^2 = 682626 = 2 \times 3 \times 7 \times 16253.$$

- $127 \equiv 1 \mod 42$, so 127 is a square modulo 2, 3 and 7.
- $541^2 \equiv 127 \mod 16253$, so 127 is a square modulo 16253.
- $\mathbb{Q}(\sqrt{127})$ has class number 1.

Let $q \in \{2, 3, 7, 16253\}$. There exists $x_q + y_q\sqrt{127}$ with norm q.

- $x_2 = 2175, y_2 = 193$ $\quad |2175^2 - 127(193)^2| = 2.$
- $x_3 = 293, y_3 = 26$ $\quad |293^2 - 127(26)^2| = 3.$
- $x_7 = 45, y_7 = 4$ $\quad |45^2 - 127(4)^2| = 7.$
- $x_{16253} = 2325, y_{16253} = 206$ $\quad |2325^2 - 127(206)^2| = 16253.$

Solution to $k = 7$

$$(45 + 4\sqrt{127})(293 + 26\sqrt{127})(2175 + 193\sqrt{127})(2325 + 206\sqrt{127})$$
$$= 533462754763 + 47337164797\sqrt{127}$$

Let

$$x = 533462754763, \qquad y = 47337164797.$$

Then

$$x^2 - 127y^2 = 682626.$$

We want to solve $2m + 1 = x$ and $2n + 127 = y$.

$$m = 266731377381, \qquad n = 23668582335.$$

$$\Delta_n + \Delta_{n+1} + \cdots + \Delta_{n+126} = \Delta_m.$$

Cohen-Lenstra Heuristics

Conjecture

Let $\mathcal{P}$ be the set of prime numbers p satisfying

1. $p \equiv 7 \mod 24$
2. $p^2 + p - 3$ *is not divisible by any prime q for which* $p \mod q$ *is a nonsquare*
3. $\mathbb{Q}(\sqrt{p})$ *has class number 1.*

The proportion of such primes is 75.45%.

This suggests

$$K(x) \gg \frac{x}{\log^{3/2}(x)}.$$

Thank you!