Underhand Free Throw Riddler Solution

Enrique Treviño

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Problem 1. Consider the following simplified model of free throws. Imagine the rim to be a circle (which well call C) that has a radius of 1, and is centered at the origin (the point $(0,0)$). Let V be a random point in the plane, with coordinates X and Y, and where X and Y are independent normal random variables, with means equal to zero and each having equal variance think of this as the point where your free throw winds up, in the rims plane. If V is in the circle, your shot goes in. Finally, suppose that the variance is chosen such that the probability that V is in C is exactly 75 percent (roughly the NBA free-throw average).

But suppose you switch it up, and go granny-style, which in this universe eliminates any possible left-right error in your free throws. Whats the probability you make your shot now? (Put another way, calculate the probability that $|Y| < 1$.)

Solution 1. Let σ be the standard deviation for X and Y. Let's find σ first. We know that the probability that $X^2 + Y^2 < 1$ is 0.75. But let's calculate the probability that $X^2 + Y^2 < 1$ in terms of σ. Since X and Y are normal variables with mean 0 and standard deviation σ, then the probability that $X < x$ is

$$P(|X| \leq x) = \frac{2}{\sqrt{2\pi}\sigma} \int_0^x e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t.$$

If $X^2 + Y^2 < 1$, then $|Y| < \sqrt{1 - X^2}$. After fixing X, we can find the probability that $P(|Y| < \sqrt{1 - X^2})$ in terms of X. The problem is that X doesn't move uniformly, so we have to figure out how to integrate this information. Let's create a Riemann Sum by cutting the interval $(0,1)$ in four pieces. We have that an approximation of the probability that $X^2 + Y^2 < 1$ is four times the following:

$$\left(\frac{1}{\sigma\sqrt{2\pi}}\int_0^{1/4} e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t\right)\left(\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\sqrt{1-x^2}} e^{-\frac{s^2}{2\sigma^2}}\,\mathrm{d}s\right) + \left(\frac{1}{\sigma\sqrt{2\pi}}\int_{1/4}^{1/2} e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t\right)\left(\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\sqrt{1-x^2}} e^{-\frac{s^2}{2\sigma^2}}\,\mathrm{d}s\right) +$$
$$\left(\frac{1}{\sigma\sqrt{2\pi}}\int_{1/2}^{3/4} e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t\right)\left(\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\sqrt{1-x^2}} e^{-\frac{s^2}{2\sigma^2}}\,\mathrm{d}s\right) + \left(\frac{1}{\sigma\sqrt{2\pi}}\int_{3/4}^{1} e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t\right)\left(\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\sqrt{1-x^2}} e^{-\frac{s^2}{2\sigma^2}}\,\mathrm{d}s\right).$$

Therefore (after multiplying by 4), we have,

$$\frac{2}{\sigma^2\pi}\left(\int_0^{1/4} e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \int_0^{\sqrt{1-x^2}} e^{-\frac{s^2}{2\sigma^2}}\,\mathrm{d}s + \ldots + \int_{3/4}^{1} e^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \int_0^{\sqrt{1-x^2}} e^{-\frac{s^2}{2\sigma^2}}\,\mathrm{d}s\right).$$

If we cut into n pieces instead of 4, we have

$$\frac{2}{\sigma^2\pi}\sum_{i=0}^{n-1}\int_{i/n}^{(i+1)/n} e^{-\frac{t^2}{2\sigma^2}}\int_0^{\sqrt{1-\left(\frac{i+1}{n}\right)^2}} e^{-\frac{s^2}{2\sigma^2}}\,\mathrm{d}s\,\mathrm{d}t.$$

To translate the Riemann Sum into an integral, we need to figure out how the integral from i/n to $(i+1)/n$ changes as $n \to \infty$. Let

$$f(x) = \int_{i/n}^{x} e^{-\frac{t^2}{2\sigma^2}}\mathrm{d}t.$$

Then $f(i/n) = 0$ and

$$f'(x) = e^{-\frac{x^2}{2\sigma^2}}.$$

By Taylor series, $f(x) \approx f(i/n) + f'(i/n)\Delta x = 0 + e^{-\frac{x^2}{2\sigma^2}}\Delta x$. Therefore the probability that $X^2 + Y^2 < 1$ is

$$\frac{2}{\sigma^2\pi}\int_0^1\int_0^{\sqrt{1-x^2}} e^{-\frac{x^2}{2\sigma^2}}e^{-\frac{y^2}{2\sigma^2}}\,\mathrm{d}y\,\mathrm{d}x = \frac{2}{\sigma^2\pi}\int_0^1\int_0^{\sqrt{1-x^2}} e^{-\frac{x^2+y^2}{2\sigma^2}}\,\mathrm{d}y\,\mathrm{d}x.$$

We're integrating over the circle of radius 1. We can change the integral to polar coordinates and get

$$\begin{aligned}\frac{2}{\sigma^2\pi}\int_0^{\pi/2}\int_0^1 e^{-\frac{r^2}{2\sigma^2}}r\,\mathrm{d}r\,\mathrm{d}\theta &= \frac{2}{\sigma^2\pi}\int_0^{\pi/2}\left((-\sigma^2)e^{-\frac{r^2}{2\sigma^2}}\right)\Big|_0^1\,\mathrm{d}\theta\\ &= \frac{2}{\pi}\int_0^{\pi/2}\left(1 - e^{-\frac{1}{2\sigma^2}}\right)\,\mathrm{d}\theta\\ &= 1 - e^{-\frac{1}{2\sigma^2}}.\end{aligned}$$

We have that the probability is 0.75, therefore we have

$$\begin{aligned}1 - e^{-\frac{1}{2\sigma^2}} &= \frac{3}{4}\\ e^{-\frac{1}{2\sigma^2}} &= \frac{1}{4}\\ -\frac{1}{2\sigma^2} &= \ln\left(\frac{1}{4}\right)\\ \frac{1}{2\sigma^2} &= \ln(4)\\ \sigma &= \sqrt{\frac{1}{2\ln(4)}}.\end{aligned}$$

Gien that value of σ, the probability that $|Y| < 1$ is

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{t^2}{2\sigma^2}} \, \mathrm{d}t \approx 0.904109.$$