Counting Perfect Polynomials

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joint work with U. Caner Cengiz and Paul Pollack

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Perfect Numbers

\( n \) is perfect if \( n \) is the sum of its proper divisors, i.e.

\[
    n = \sum_{d|n, d \neq n} d
\]

Examples:

\[
    6 = 1 + 2 + 3 \\
    28 = 1 + 2 + 4 + 7 + 14 \\
    496 = 1 + 2 + 4 + 8 + 16 + 31 + 31 \cdot 2 + 31 \cdot 4 + 31 \cdot 8 \\
    2^{p-1} (2^p - 1) = 1 + 2 + 4 + \ldots + 2^{p-1} + (2^p - 1) \left( 1 + 2 + 4 + \ldots + 2^{p-2} \right)
\]

for \( 2^p - 1 \) prime (i.e., a Mersenne prime).
A polynomial mod 2 is one of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_i \in \{0, 1\}$.

We consider the operation mod 2, i.e.,

$1 + 1 = 0, 0 + 1 = 1 + 0 = 1, 0 + 0 = 0$.

For example

$$x^2 + 1 = x^2 + 2x + 1 = (x + 1)^2.$$
Let $\sigma(P)$ be the sum of the divisors of a polynomial $P$ in mod 2.

A polynomial is said to be perfect mod 2 if $\sigma(P) = P$.

$x^2 + x = x(x + 1)$, so

$$\sigma(x^2 + x) = 1 + x + (x + 1) + x^2 + x = x^2 + x.$$ 

So $x^2 + x$ is perfect.

$$\sigma(x^2 + 1) = 1 + (1 + x) + (1 + x^2) = 1 + x + x^2,$$

so $x^2 + 1$ is not perfect.
Let $P(x) = (x(x + 1))^{2^n - 1}$. We’ll show $\sigma(P) = P$.

\begin{align*}
1 + x + x^2 + \cdots + x^{2^n - 1} &= \frac{x^{2^n} - 1}{x - 1} = \frac{x^{2^n} + 1}{x + 1} = (x + 1)^{2^n - 1}.

1 + (1 + x) + \cdots + (1 + x)^{2^n - 1} &= \frac{(1 + x)^{2^n} - 1}{x} = \frac{1 + x^{2^n} - 1}{x} = x^{2^n - 1}.

\sigma(P) = \sigma(x^{2^n - 1})\sigma((x + 1)^{2^n - 1}) = (x + 1)^{2^n - 1} \cdot x^{2^n - 1} = P.
\end{align*}
### Weirdo Perfects

<table>
<thead>
<tr>
<th>Degree</th>
<th>Factorization into Irreducibles</th>
</tr>
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<tbody>
<tr>
<td>5</td>
<td>$T(T+1)^2(T^2+T+1)$</td>
</tr>
<tr>
<td></td>
<td>$T^2(T+1)(T^2+T+1)$</td>
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<tr>
<td>11</td>
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<tr>
<td></td>
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<td>$T^4(T+1)^3(T^4+T^3+T^2+T+1)$</td>
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<tr>
<td>15</td>
<td>$T^3(T+1)^6(T^3+T+1)(T^3+T^2+1)$</td>
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<td>16</td>
<td>$T^4(T+1)^4(T^4+T^3+1)(T^4+T^3+T^2+T+1)$</td>
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<tr>
<td>20</td>
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</tr>
<tr>
<td></td>
<td>$T^6(T+1)^4(T^3+T+1)(T^3+T^2+1)(T^4+T^3+1)$</td>
</tr>
</tbody>
</table>

**Figure:** Perfect numbers not in the infinite family. Found by Canaday in 1941
We say that $P$ is an even perfect if $x(x + 1)|P$ and $P$ is perfect.

We say that $P$ is odd otherwise.

Conjecture

All perfect polynomials are EVEN.
What did we know

**Theorem (Canaday)**

An odd perfect polynomial is a square.

**Theorem (Gallardo-Rahavandrainy)**

If $A$ is an odd perfect polynomial, then it has at least 5 distinct irreducible factors. Moreover, the number of irreducible factors of $A$, counted with multiplicity, is at least 12.
What did we prove

**Theorem (Cengiz-Enrique-Pollack)**

*The number of perfect polynomials of norm \( \leq x \) is \( O_\epsilon(x^{1/12+\epsilon}) \) for every \( \epsilon > 0 \).*

The norm of \( A \) is \( 2^{\deg A} \).

**Theorem (Cengiz-Enrique-Pollack)**

*There are no odd perfect polynomials of degree \( \leq 200 \), i.e., there are no odd perfect polynomials of norm \( \leq 2^{200} \approx 1.6 \times 10^{60} \).*

**Theorem (Cengiz-Enrique-Pollack)**

*If \( A \) is a non-splitting perfect polynomial of degree \( \leq 200 \), then \( A \) is one of Canaday’s polynomials.*
Lemma (Fundamental lemma)

Let $M$ be a polynomial which is not perfect, and let $k \geq 2$ be a fixed positive integer. Let $x \geq 10$. Then there exists a constant $C_k$ depending only on $k$, as well as a set $S$ depending only on $M$, $k$ and $x$, of cardinality bounded by $x^{C_k/\log \log x}$, with the following property: if $A$ is a perfect polynomial of norm $\leq x$ for which

(a) $M$ is a unitary divisor of $A$: i.e., $A = MN$ with $\gcd(M, N) = 1$, and

(b) $N = A/M$ is $k$-free, i.e., $P^k \nmid N$ for any irreducible polynomial $P$, then $A$ has a decomposition of the form $M'N'$, where

1. $M'$ is an element of $S$,
2. $M'$ and $N'$ are unitary divisors of $A$,
3. both factors $M'$ and $N'$ are perfect,
4. $N'$ is $k$-free,
5. $M$ is a unitary divisor of $M'$. 
H.-W. Algorithm

Given a polynomial $B$ and a stopping bound $H$, with $\deg B \leq H$, the following algorithm (a) outputs only perfect polynomials $A$ of degree $\leq H$ having $B$ as a unitary divisor, and (b) outputs every such $A$ that is indecomposable.

1. Check if $\sigma(B) = B$. If yes, then output $B$ and break.
2. Compute $D = \sigma(B) / \gcd(B, \sigma(B))$.
3. If $\gcd(B, D) \neq 1$, break.
4. Let $P$ be an irreducible factor of $D$ of largest degree.
5. Recursively call the algorithm with inputs $BP^k$ and stopping bound $H$, for all positive integers $k$ with $\deg(BP^k) \leq H$.

Note: indecomposable means $A$ has no nontrivial factorization as a product of two relatively prime perfect polynomials.
Recursion

Figure: Recursion for the Algorithm
How did we go so high?

To check odd perfects:
- From the algorithm, we need only check whether $P^2$ is a unitary divisor for $\deg P \leq 20$.
- Because if $A$ is perfect. It has at least 5 prime divisors and $A$ is a square.

To check even perfects that are not in the infinite family:
- If $P(x)$ is perfect, then $P(x + 1)$ is perfect.
- If $P$ is perfect $x|P \iff (x + 1)|P$.
- We need only check the algorithm for $x, x^2, \ldots x^{100}$.
Thank you!