Playing with Triangular Numbers

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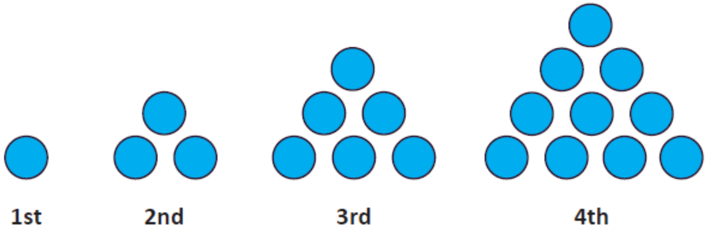
Triangular Numbers

What are triangular numbers?



Triangular Numbers

The n-th triangular number, Δ_n is $\dfrac{n(n+1)}{2}$

Combinatorial Proof:

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n}\sum_{j=0}^{i-1} 1 = \#\{(i,j) \mid 0 \le j < i \le n\} = \binom{n+1}{2}.$$

Probabilistic Proof: Let X be the sum of two dice.

$$\mathbb{P}[X=2] = \frac{1}{6^2},\ \ \mathbb{P}[X=3] = \frac{2}{6^2}, \cdots, \mathbb{P}[X=7], = \frac{6}{6^2},$$
$$\mathbb{P}[X=8] = \frac{5}{6^2}, \cdots, \mathbb{P}[X=12] = \frac{1}{6^2}.$$

$$\frac{1+2+\cdots+6+5+\cdots+1}{6^2} = \frac{2(1+2+\cdots+6)-6}{6^2} = 1.$$

Playing with Triangular Numbers

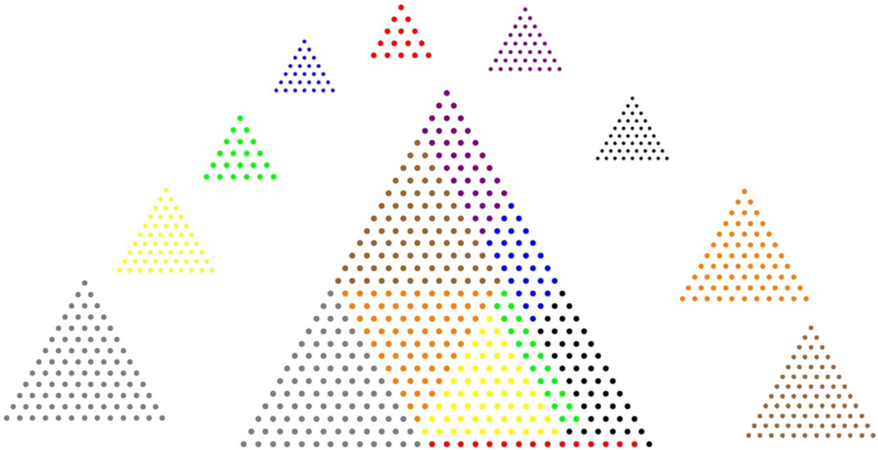
$$1 + 3 + 6 = 10$$

McMullen, inspired by this, asked himself:

- For which k can we find k consecutive triangular numbers that add up to be a triangular number?
- Can we find the solutions?

McMullen showed there are infinitely many solutions for $k = 2, 3, 5$, but no solutions for $k = 4$.

Example



$k = 4$ case

Elementary manipulations show that the sum of the k-consecutive triangular numbers starting at Δ_n is Δ_m whenever

$$(2m+1)^2 - k(2n+k)^2 = \frac{(k-1)(k^2+k-3)}{3}.$$

When $k = 4$ we get

$$(2m+1-4n-8)(2m+1+4n+8) = 17.$$

From which

$$(m,n) = (4,0), (4,-4), (-5,0), \text{ and } (-5,-4).$$

Square k

Theorem

Let $k > 4$ be a square. Then there exist k consecutive triangular numbers that add up to make a bigger triangular number.

$k = 6$

Recall
$$(2m+1)^2 - k(2n+k)^2 = \frac{(k-1)(k^2+k-3)}{3}.$$
When $k = 6$:
$$x^2 - 6y^2 = 65.$$
Therefore $x^2 \equiv 6y^2 \bmod 13$. But
$$\left(\frac{6}{13}\right) = -1.$$
Therefore, there are no solutions for $k \equiv 6 \bmod 13$.

A sufficient condition for k

Lemma

Let $q > 3$ be a prime number. Suppose that $k \in \mathbb{Z}$ is such that

1. *k is not a square modulo q,*
2. *$q \parallel k^2 + k - 3$.*

Then there are no k consecutive triangular numbers that add up to a triangular number.

Example:

If $k \equiv 45 \mod 53$ and $k \not\equiv 2430 \mod 53^2$. There are 52 residues modulo 53^2 which satisfy these conditions.

Finding q such that $q \parallel k^2 + k - 3$ and $(k/q) = -1$

If $q \neq 13$, $k^2 + k - 3 \equiv 0 \bmod q$ has two distinct solutions k_1, k_2 whenever $(13/q) = 1$. We then have three possibilities

1. Both k_1, k_2 are squares modulo q.
2. One of k_1, k_2 is a square and the other one isn't.
3. Neither k_1, k_2 are squares modulo q.

A pair of important sets of primes

Let $\mathcal{A}$ be the set of primes q for which we have k_1, k_2 both nonsquares modulo q.

Let $\mathcal{B}$ be the set of primes q for which exactly one of k_1, k_2 is a square modulo q.

If $q \in \mathcal{A}$, then the proportion of residues modulo q^2 one must avoid are

$$2\frac{q-1}{q^2} = \frac{2}{q} - \frac{2}{q^2}.$$

If $q \in \mathcal{B}$, then the proportion of residues modulo q^2 one must avoid are

$$\frac{q-1}{q^2} = \frac{1}{q} - \frac{1}{q^2}.$$

Quantifying the proportion of primes in $\mathcal{A}$, $\mathcal{B}$

Consider $f(x) = x^4 + x^2 - 3$. Let's analyze how $f(x)$ might factor in $\mathbb{Z}_q$. There are several possibilities

- (1,1,1,1)
- (1,1,2)
- (2,2)
- 4

Primes in $\mathcal{B}$ would split as (1,1,2).

Primes in $\mathcal{A}$ would be primes that are squares modulo 13 and that don't split as (1,1,2) or (1,1,1,1).

Chebotarev in action

Consider $f(x) = x^4 + x^2 - 3$. f is irreducible over $\mathbb{Q}$, let $\mathbb{L}$ be the splitting field of f over $\mathbb{Q}$, then $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ is isomorphic to

$$\{(1),(1324),(12)(34),(1423),(34),(13)(24),(12),(14)(23)\}.$$

- 1 of the 8 elements decompose as (1,1,1,1)
- 3 of the 8 elements decompose as (2,2)
- 2 of the 8 elements decompose as (1,1,2)
- 2 of the 8 elements decompose as (4)

The proportion of primes $q \in \mathcal{B}$ is $2/8 = 1/4$.

The proportion of primes $q \in \mathcal{A}$ is $1/2 - 1/8 - 2/8 = 1/8$.

Idea of Proof

There are several residues modulo certain squares of primes that must be avoided for k to be able to yield solutions.
We then get the following upper bound heuristic:

$$K(x) \ll x \prod_{\substack{q \leq x \\ q \in \mathcal{A}}} \left(1 - \frac{2}{q} + \frac{2}{q^2}\right) \prod_{\substack{q \leq x \\ q \in \mathcal{B}}} \left(1 - \frac{1}{q} + \frac{1}{q^2}\right)$$

$$\ll x \left(\frac{1}{(\log^{1/8} x)^2}\right) \left(\frac{1}{\log^{1/4}(x)}\right)$$

$$\ll \frac{x}{\sqrt{\log x}}.$$

Main Theorem

Theorem

Let $K(x)$ be the number of k's less than x that have solutions. Then:

$$\sqrt{x} \leq K(x) \ll \frac{x}{\sqrt{\log(x)}}.$$

$k = 127$

We want to solve

$$(2m+1)^2 - 127(2n+127)^2 = 682626 = 2 \times 3 \times 7 \times 16253.$$

- $127 \equiv 1 \mod 42$
- $541^2 \equiv 127 \mod 16253$
- $\mathbb{Q}(\sqrt{127})$ has class number 1.

Let $q \in \{2, 3, 7, 16253\}$. There exists $x_q + y_q\sqrt{127}$ with norm q.

- $x_2 = 2175, y_2 = 193$
- $x_3 = 293, y_3 = 26$
- $x_7 = 45, y_7 = 4$
- $x_{16253} = 2325, y_{16253} = 206$

Solution to $k = 127$

$$(45 + 4\sqrt{127})(293 + 26\sqrt{127})(2175 + 193\sqrt{127})(2325 + 206\sqrt{127})$$
$$= 533462754763 + 47337164797\sqrt{127}$$

Let

$$x = 533462754763, \qquad y = 47337164797.$$

Then

$$x^2 - 127y^2 = 682626.$$

We want to solve $2m + 1 = x$ and $2n + 127 = y$.

$$m = 266731377381, \qquad n = 23668582335.$$

$$\Delta_n + \Delta_{n+1} + \cdots + \Delta_{n+126} = \Delta_m.$$

On the hunt for a lower bound

Our goal is solving

$$(2m+1)^2 - k(2n+k)^2 = \frac{(k-1)(k^2+k-3)}{3}.$$

Let p be a prime number satisfying:

1. $p \equiv 7 \bmod 24$
2. $p^2 + p - 3$ is not divisible by any prime q for which $p \bmod q$ is a nonsquare
3. $\mathbb{Q}(\sqrt{p})$ has class number 1.

Then there exist p consecutive triangular numbers that add up to a triangular number.

Cohen-Lenstra Heuristics

Conjecture

Let $\mathcal{P}$ be the set of prime numbers p satisfying

1. $p \equiv 7 \mod 24$
2. $p^2 + p - 3$ *is not divisible by any prime* q *for which* $p \mod q$ *is a nonsquare*
3. $\mathbb{Q}(\sqrt{p})$ *has class number 1.*

The proportion of such primes is 75.45%.

This suggests

$$K(x) \gg \frac{x}{\log^{3/2}(x)}.$$

You're Welcome!