Walking on Rational Numbers and a Self-Referential Formula

Margaret Fortman* Kevin Kupiec†
Marina Rawlings‡ Enrique Treviño§

January 15, 2018

1 Walking on Numbers

In [1], Aragón Artacho, et al. describe the process of a walk on the plane using the digits of a number base 4.[1] Consider a number x written in base 4, $x = d_n d_{n-1} \ldots d_0 . d_{-1} d_{-2} \ldots$. We start at the origin in the Cartesian plane. If $d_n = 0$ we move a unit to the right, if it is 1, we move a unit upwards, if it is 2, we move a unit to the left, and if it is 3, we move downwards. We continue this process with $d_{n-1}, d_{n-2}, \ldots, d_1, d_0, d_{-1}, \ldots$, and this process creates a "walk." For example, the number 419636198 is rewritten in base 4 as 121000302033212_4. The walk would look like Figure 1.

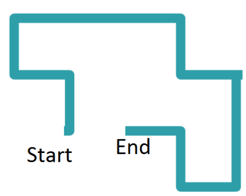

Figure 1: Walk for the number 419636198, which is 121000302033212_4.

In [1], they show several other walks, including walks on π and e using 100 billion digits. The one that inspired this paper is the following: Consider the

*Lake Forest College student, fortmanma@mx.lakeforest.edu
†Lake Forest College 17', kupieck@mx.lakeforest.edu
‡Lake Forest College 17', rawlingsmn@mx.lakeforest.edu
§Department of Mathematics and Computer Science at Lake Forest College, trevino@lakeforest.edu

[1] They describe the process in any base, but for the purposes of this paper, we'll focus on base 4.

rational number

$$
\begin{aligned}
Q = &1049012271677499437486619280565448601617567358491560876166848380843144 3\\
&5844725287555162924702775955557045371567931305878324772977202177081818 7\\
&9659063736576748798142280132859202786101925814095713574870471229026746 5\\
&151312805954195399750420206138037382233895971339195 4\\
&/\\
&1612226962694290912940490066273549214229880755725468512353395718465191 3\\
&5301734881431401750453996944547935301206438332726709700793305262920303 5\\
&0920973600450955456136596649325078391464772840162385651374295294530896 1\\
&2268152748875615658076162410788075184599421938774835.
\end{aligned}
$$

The walk on this number is Figure 2.[2]

Figure 2: Walk on Q.

In Section 2 we describe an algorithm that, given any sentence σ, it can find a rational number whose walk creates an image that spells out σ. We do something similar in Section 4, but instead of using walks as our starting point we use Tupper's self-referential formula, which is described in Section 3.

2 Writing using Walks

For each letter of the alphabet α, we wish to find a rational r whose walk produces an image of α, Just as the previous rational has produced the Q in Figure 2. How do you find the rational for a particular letter? First, find a string of digits that "spell out" your letter in such a way that you finish where you started. You can do this by hand. For example, for the letter "D", you can draw a stylish D as in Figure 3, then translate every movement as digits. In this case we obtain 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 0, 0, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 2, 2. Now, grab the first digit and multiply it by $\frac{1}{4}$, then add to it the second digit multiplied by $\frac{1}{4^2}$,

[2]This figure comes from [1]. As the authors explain, the color in this walk indicates the path followed by the walk. It is shifted up the spectrum (red-orange-yellow-green-cyan-blue-purple-red).

and so on. In our example we'd have

$$\delta = 2\left(\frac{1}{4}\right) + 1\left(\frac{1}{4^2}\right) + 1\left(\frac{1}{4^4}\right) + \ldots = \frac{6.38477\ldots \times 10^{74}}{2.32588\ldots \times 10^{74}}.^{3} \tag{1}$$

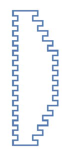

Figure 3: Walk on the rational δ creating the letter "D".

If we do a walk of δ with 124 steps, then we'll get Figure 3. However, this rational, has only 124 significant digits base 4, i.e., every digit afterwards is 0. Therefore, if you consider the "infinite walk", then it will not spell "D" anymore, it will start heading eastwards, creating an underlined "D". To fix this, we create copies of the digits that form one loop of "D" and glue them every 124 steps, which is equivalent to multiplying δ by powers of 4^{-124} and adding them up. We get the geometric series:

$$\mathcal{D} = \delta + 4^{-124}\delta + \left(4^{-124}\right)^2\delta + \ldots = \frac{1}{1-4^{-124}}\delta = \frac{4^{124}}{4^{124}-1}\delta. \tag{2}$$

The walk for $\mathcal{D}$ for any number of steps ≥ 124 will create Figure 3.

Now, suppose that for every letter α (a variable representing an uppercase letter from the English alphabet), you find a rational r_α and an integer n_α, such that the walk of r_α with n_α steps spells the letter α in such a way that the last step ends at the origin. We will also define $r_{blank} = n_{blank} = 0$, i.e., representing a blank space. Finally, suppose the base of each letter is at most w, i.e., the length of a blank space is w.

Theorem 1. *Suppose we are given a sentence σ which we'll write as $\sigma = \alpha_1\alpha_2\alpha_3\cdots\alpha_k$, where α_i is a letter or a space. Let*

$$n = \sum_{i=1}^{k} n_{\alpha_i} + 2(k-1)w, \tag{3}$$

and

$$r = \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_j}+w)\right)}} + \frac{2\cdot 4^w\left(1-\frac{1}{4^{(k-1)w}}\right)}{3\cdot 4^{\left(\sum_{j=1}^{k}(n_{\alpha_j}+w)\right)}}. \tag{4}$$

Then a walk on r of length n spells out σ. Furthermore, a walk on $\frac{4^n}{4^n-1}r$ of any length $m \geq n$ spells out σ.

[3] Because of the lentgh of many numbers in this paper, we will restrict most of them to their first six digits. In the case of fractions we also keep the power of 10 to show how many digits a number has. The complete numbers can be found in the Appendix.

Proof. The idea is to focus on the digits first. We have n_{α_1} digits to represent α_1, then we include w zeroes to give space for the next letter. We follow this with n_{α_2} digits of the second letter, followed by w zeroes, and so on. When we "write" the last letter, we have used $n_{\alpha_1} + n_{\alpha_2} + \ldots + n_{\alpha_k} + (k-1)w$ digits. But the walk is $(k-1)w$ steps to the right. To get back to the origin, we need to take $(k-1)w$ steps to the left, i.e., we need $(k-1)w$ 2's in the digit expansion. Therefore we've used n digits where n is the same as in (3).

Now we want to find the rational that has this digit expansion. To account for the letter α_i in the desired position, we need to multiply it by 4^{-x} where x is the number of digits used so far. For α_1, we've used 0, for α_2 we've used $n_{\alpha_1}+w$, for α_3 we've used $(n_{\alpha_1}+w)+(n_{\alpha_2}+w)$, and in general for α_i, we've used $\sum_{j=1}^{i-1}(n_{\alpha_j}+w)$. Finally, we have to take into account the final $(k-1)w$ 2's. To do this, we can think of $r_{\alpha_{k+1}} = 0.22\cdots 2_4$ and place it after all the digits so far, which have been $\sum_{i=1}^{k-1}(n_{\alpha_i}+w) + n_{\alpha_k}$. Therefore

$$\begin{aligned} r &= \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_i}+w)\right)}} + \frac{r_{\alpha_{k+1}}}{4^{\left(\sum_{i=1}^{k}(n_{\alpha_i}+w)-w\right)}} \\ &= \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_i}+w)\right)}} + \frac{4^w}{4^{\left(\sum_{i=1}^{k}(n_{\alpha_i}+w)\right)}} \left(\frac{2}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \ldots + \frac{2}{4^{(k-1)w}}\right). \end{aligned}$$

By completing the geometric series we can verify this matches (4). By construction, the walk for r with n steps spells out σ. Furthermore, by the same process as that in (2), we find the rational whose infinite walk spells out σ. □

Theorem 1 suggests how to build a program to find a rational number for any sentence. As an example, a certain rational

$$r_\sigma = \frac{3.47783\ldots \times 10^{3195}}{5.42542\ldots \times 10^{3195}}, \tag{5}$$

creates Figure 4.

Figure 4: The walk for r_σ as in (5) after 10000 steps.

3 Tupper's self-referential formula

In [4], Tupper introduced the formula[4]

$$\frac{1}{2} < \left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \mathrm{mod}(\lfloor y \rfloor, 17)}, 2\right)\right\rfloor. \tag{6}$$

[4]The formula was given as an example of a formula that graphing software had difficulties with, but Tupper's graphing software can handle.

This formula has the amazing property that if you graph the equation[5] for $0 \le x < 106$ and $k \le y < k + 17$ for k as in (7) [6] you get Figure 5.

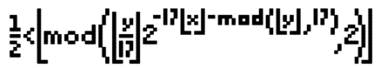

Figure 5: Graph of (6) in the range $0 \le x < 106$ and $k \le y < k + 17$.

$$\begin{aligned} k =& 4858450636189713423582095962494202044581400587983244549483093085061934704708809 \\ & 9284506447698655243648499972470249151191104116057391774078569197543265718554420 \\ & 5721044573588368182982375413963433822519945219165128434833290513119319995350241 \\ & 3758765239264874613394906870130562295813219481113685339535565290850023875092856 \\ & 8926945559742815463865107300491067230589335860525440966643512653493636439571255 \\ & 6569593681518433485760526694016125126695142155053955451915378545752575659074054 \\ & 0157929001765967965480064427829131488548259914721248506352686630476300. \end{aligned} \tag{7}$$

It turns out that the formula doesn't only graph itself, by considering different values of k, we can graph anything that can be represented by pixels in a 106×17 table. For example, a certain value

$$k_0 = 1.4452 \ldots \times 10^{536} \tag{8}$$

gives the interval in which the graph looks like Figure 6.

Figure 6: Graph of (6) in the range $0 \le x < 106$ and $k_0 \le y < k_0 + 17$.

The main reason why we can build anything in a 106×17 grid is the following lemma:

Lemma 1. *Let $k = 17k'$ for a nonnegative integer $k' < 2^{106 \times 17}$. Suppose we write k' in binary as follows:*

$$k' = \sum_{m=0}^{105} \sum_{n=0}^{16} a_{17m+n} 2^{17m+n}. \tag{9}$$

Then

$$\left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \mathrm{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor = a_b, \tag{10}$$

for $b = 17\lfloor x \rfloor + \mathrm{mod}(\lfloor y \rfloor, 17)$.

Therefore, the point (x, y) is painted whenever $a_b = 1$ and not painted when $a_b = 0$, i.e., it depends only on the binary expansion of k'.

[5]By this we mean that the point (x, y) is painted black if it satisfies the inequality and not painted if it doesn't.

[6]In [2] and many other places, the value of k is given differently because of the convention in computer science that positive y go downwards.

Proof. Let $\lfloor x \rfloor = i$ and $\lfloor y \rfloor = j = 17k' + j'$ for some $0 \le j' \le 16$. Then $\left\lfloor \frac{y}{17} \right\rfloor = k'$, $j' = \mathrm{mod}(\lfloor y \rfloor, 17)$, and $b = 17i + j'$. Now

$$\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \bmod (\lfloor y \rfloor, 17)} = k' 2^{-17i - j'} = \sum_{m=0}^{105} \sum_{j=0}^{16} a_{17m+n} 2^{17m+n-17i-j'}.$$

When we consider mod 2, we can eliminate any term where the exponent of 2 is at least 1, i.e., we're left with exponents satisfying $17m + n - 17i - j' \le 0$. When we take the floor, we exclude any of the small exponents because $1/2 + 1/4 + \ldots + 1/2^c < 1$ for any finite c. Therefore the only exponent of 2 we allow is 0. Hence $17m + n - 17i - j' = 0$. This implies $n \equiv j' \bmod 17$, but both n and j' are between 0 and 16, so $n = j'$, and then $m = i$, which is what we wanted to prove. □

4 Writing using Tupper's self-referential formula

From Lemma 1 we can extrapolate an algorithm to find a k to build any picture in a 106×17 grid. Indeed, write a 1 on any unit square that is painted black and a 0 otherwise. Now starting at the square with bottom-left corner $(0, 0)$, read the digits from bottom to top on each column. This binary number (read from right to left) will be k' and so $k = 17k'$.

Problem: For a given sentence, find the integer k such that the graph of Tupper's formula looks like that sentence for $0 \le x < 106$ and $k \le y < k + 17$.

As in the walk example, the key is figuring out how to do a letter first. Let's demonstrate how to do the letter a. Consider Figure 7. We read the number as 11101 10101 11111. To transform it into a number that fits in the 106×17 grid, we need to fill in the necessary 0's, which is equivalent to multiplying numbers in the ℓ-column by $2^{17(\ell-1)}$. Therefore, we associate the letter "a" with the number

$$17\left((1+2+4+16) + (1+4+16)2^{17} + (1+2+4+8+16)2^{34}\right). \qquad (11)$$

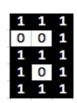

Figure 7: Breaking down the letter "a" in binary.

We can now move a letter around the grid by multiplying it by 2^{17m+n} to place it where the bottom-left corner of the letter is (m, n). If we create all letters with a height of at most 5 squares and width of at most 5 squares (the letters "m" and "w" need 5 squares, and the rest need 3), we can then fit up to three rows of letters to spell a short sentence. Given a letter α, let $f(\alpha)$ be the number we associate with α with bottom-left corner on $(0, 0)$. We'll let $f_{blank} = 0$ and for letters with width 3, we'll multiply their numbers by 2^{17},[7] to create a buffer between letters.

Theorem 2. *Given a sentence $\sigma = \alpha_1\alpha_2 \cdots \alpha_k$, where α_i represents a single letter or a blank space and $k \leq 63$, we use the following formula to figure out the value of k for the range where the plot of Tupper's formula is σ:*

$$\sum_{i=1}^{\min(21,k)} 2^{85(i-1)+12} f(\alpha_i) + \sum_{i=22}^{\min(42,k)} 2^{85(i-22)+6} f(\alpha_i) + \sum_{i=43}^{k} 2^{85(i-43)} f(\alpha_i). \quad (12)$$

Proof. Each letter fits in a block of width 5 and height 5. To move from one letter to the next (to the right), we need to multiply by $2^{17\times5} = 2^{85}$. This is where the 85's in the exponents come from. The reason we add 12 and 6 (depending on how many letters we have) is because the first row consists of numbers in the top "strip" $(k + 12 \leq y < y + 17)$, so we have to multiply by 2^{12} to move upwards. The numbers in the middle strip $(k + 6 \leq y \leq y + 11)$ need a shift of 2^6, and the bottom row needs no translation. The formula follows. □

As an example of finding a k for a particular phrase, Figure 8 is the plot of Tupper's formula for $0 \leq x < 106$ and $k_1 \leq y < k_1 + 17$ for a certain

$$k_1 = 6.20234\ldots \times 10^{461}. \quad (13)$$

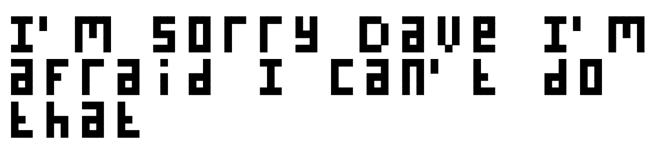

Figure 8: Graph of (6) in the range $0 \leq x < 106$ and $k_1 \leq y < k_1 + 17$.

Acknowledgements

We used Mathematica to create the walks and to plot Tupper's self-referential formula in different ranges. We would like to thank Scott T. Chapman for his suggestions that improved the exposition of this paper, and to Pete Winkler for comments on the paper. We would also like to thank the anonymous referee for their thoughtful comments.

[7] The only letters not multiplied by 2^{17} are "m" and "n".

References

[1] Francisco J. Aragón Artacho, David H. Bailey, Jonathan M. Borwein, and Peter B. Borwein, *Walking on real numbers*, Math. Intelligencer **35** (2013), no. 1, 42–60.

[2] B. Haran and M. Parker, *The 'everything' formula - numberphile*, `https://www.youtube.com/watch?v=_s5RFgd59ao`, Accessed: 2017-05-23.

[3] Wolfram Research, Inc., *Mathematica, Version 10.0*, Champaign, IL, 2017.

[4] Jeff Tupper, *Reliable two-dimensional graphing methods for mathematical formulae with two free variables*, Proceedings of the 28th Annual Conference on Computer Graphics and Interactive Techniques (New York, NY, USA), SIGGRAPH '01, ACM, 2001, pp. 77–86.

5 Appendix: Full decimal digit expansion of constants in the paper.

The value of δ in (1) is

$$\begin{aligned}\delta =&6384779382043951036217348661253680515005885357484535471589654514956414794662\\&72100636854259724898698532312741670451981081526131897015418 3\\&/\\&2325883917745942049757836185241614509931652354199417792900768637378045721962\\&873354643811362284043409794440069140051769387310725211566899 2.\end{aligned} \tag{14}$$

The value of k_0 in (8) is

$$\begin{aligned}k_0 =&1445202489708975828479425373371945674812777822151507024797188139685490\\&8873568298734888825132090576643817888323197692344001666776474924212512\\&8995265907053708020473915320841631792025549005418004768657201699730466\\&3833949016013743197155209961811452497819450190683595005106578043256408\\&0119786755686314228025969420625409608166564241736740394638417077453742\\&7319606443899923010379398938675025786929455234476319291860957618345432\\&2480049217280333494198162067498544720381939397385138489604767597826733\\&1343769705199458068186981933044633677404726886 4,\end{aligned} \tag{15}$$

The value of k_1 in (13) is

$$\begin{aligned}k_1 =&6202342045523518696372190728132145377913289497819263812843155643364944\\&0461939615515996102292719598762206682015817244444562918664906697777197\\&5007949955351598702405129648571930754026169504789347614953307622064658\\&7662203381308047340029024837030531000814297140117523848644113896733785\\&5616409282734138890208876466646383764272086299397454808405688789312744\\&7832949435883695715278636348898143061593729742606126050532003884145813\\&57448085400074739752361379659227287086694 4.\end{aligned} \tag{16}$$

The value of r_σ in (5) is

$r_\sigma =$

3477838176632809766274027652998361070109305198798526822993346685629729543816284212324168906789690775547148059136594
5650708415322762044997606235997855677049782095397960796362753436599841518280673260542696144332265810919906062866086
7300310028185786493083481099698081973220156381116333517929438873090176653898133176194667067667046684812642590682921
7676064926655968063069943083237148802645189802820278834066659917243976811957017195714300536678576755193478646225748
0029718138397905599265918142472752199419439333830496840583774337111547232356603947260968981693338103053145854635253
6556487569357115640975974485570693172272764417938367108726707881765559137578753568579282574517818911608998412741042
0227212620869607503190020705525648927347045787545010799076183740241404923921318243481736923524124663788869995658064
9792376238237930407962450320389159643655424259838617670381363425821742408854318599877016505639048938102659648790392
9595397634307377756861919915276546513631911373910771937588542354894963935837136025758958018813262204515835086096680
5708164505804035890264351831753362652746839651784749459769618297620283942247979473521463646695449617575428923630172
0483591538711675276566969776077077093216794342308880360116044571717992965087557491510480694730625476830869629084918
0330152435295444522317092256763450131932664980244104376549124516719713135304140238008592647538068871086358474426810
3228839275062729870736963264331850092569189868395203739083590687341840855847621655276649131461408882329507967011133
9673513989596387755006734087403631374900979733345825183520000390763148149610355789986385442991553413321157523278681
4570776631231717821839613245063461887765374214685496293414305359730563242820842921560948381487459668207389426718341
0208201125760638630177927567438294560420792809301856500683312377648424972104939732771020518033666376045372522152359
6365323093742860258648736509758715341472162499794997150383376599167041645495565423248191931830700401690739885894446
5925138828780782529332465419011534533168166732703212020724420330540765914808719296089874268534962295074250221700319
4354384010924295515836192696117735836407353432915120743106575043357828096478475426584678135447571530015321875975666
0288499555262323407672275144428184247080572013967367202580538705898188934186489884901842932385285873056974563406043
2972003424454777927419274342586200406369883986949725285622098211132577303257888674921256677207569164691929862742553
6434760361740959640882424911067801423858038854691012806455532141473569316256449253261282819184164210728465538046128
8689316433179357074775456515481274301511980843961573141008058309365269614398459837204683190833300361309048467141298
7077183501524879707368429696020266759574384790866466734735759122149901064852517874055671305612116400142877895886450
1717055895264820300249217342710966276085484922766003610831279480622647998891816267884673999729550256532945744911484
6457331356310576117617522162604119512965408010784378997607700863451724450862932076743527823981076515112164422001921
6801892221050493956203183395973148259212386859598795634512915298898905336948973956044452272241532431935859131543491
8841290611039546092076535887617550365638779065363723042885225828369392365302807084634122922

/

5425427555547183235387483138677443269370516110125718034418458072780400020660097464368420331167768059210326095280700
7960754517818980320913237680771539405202704824855426057414759118003163672774854142023984371664371147816596771028104
6441014848078122087947108444254899832081659029137741969600347998581332358020366378254707477828257293605011341272917
7555747527995632047517011038746128120784347825211963664998369471837665640062502456285346080810630834125801925184520
2625907527220125563925851141313331030922107272915406124790182288452890603674783060337942569320897590240195578623903
9566157200553376651298625999377015571387110468470489015343875859756592220376633541514211837269398010309207318883562
6661594155820595537918980228922993377013997290067594729152329146585615037257856776126270118053463580693713572941709
0234335508300132571024621930514890212320934438015747210297144734254112822232392753999420252062509544044913507097587
9350699695390168912767463858081679059061108453821615559680371903283785451415371400938993626727592450435669695632669
7254222479018247794025921399900915178303441159892932368904241515358090051156613040271864216857817861614058082054879
7992961270430349541616758800746625539427565309875638214927919257745396009984319529064787174938772005587710967554787
3134637767111202690659149688125290400377854774888701800589697431906307040676654222644000854136099592624341159355680
4025055285850722385200868824064066076467668623572504966498893887720846592451499855656864232341936775024254014988573
6479995630540608861215015614298371308536980538926684840420124076706834851544084593920349240191074869574272723680521
7127047049869920301330050108161034628412359529481374507528437783717961628384774969135064191568538488550802164466946
2991844172716763186432074166709520943957216137007438680496257872185964930482173472798553657122383518454992612482965
8983430312485656075937606211310154685225690149806271811259712611252549777641482169337757296336319039098051110001043
7499227486471556829577571187480183096203689144604478969275431212534506308782256889985344653287072554683775069061486
3014073442952271232361355689066831264378429515802977509102274184384658882941102930608058230829056588956526453972913
5174248235942763565173944771076327253058869928258224619285181326402922888853044020007731096127030815289259831325582
5255354353810146217297245345948544438608118775507870315659930862311032719187735199457377847675540579010031134555663
4489008835672563900761318061355322849160270301707208801750032777202061389610811502156139259718144929520269648838872
0466980806627099676908203772701028988981196334425767237037978129102064210173720253085972283392852129043684301723824
2526468727429343973246435674499354269034727279971334516277822171422775378729827488841283446462926242824176152611082
7525355700871671055020654492230456283388123851578103617271263271404921176203093075586296285974629333519569598503406
2164519084107133103865309775307272676431232643151623368126225771314148471748923659994901815532321569291863200955776
1945338825314444239627509164817263099215974450756080971015744259843605203901279813501743110877702438536565400720937
3657055236584191717148472650721961987021467849755631867227974194334896125257314027790401535.