Expected Number of Dice Rolls for the Sum to Reach n

A classic probability result is that the expected number of random summands chosen uniformly from the interval [0, 1] that are needed to sum to at least 1 is e. This result is mainstream enough that it has appeared in general audience books such as [**4**, p. 137] (a proof can be found in [**3**]). Here we consider a discrete variation, namely, let X be the number of times an n-sided die has to be rolled so that the sum of the rolls is at least n. Then

Theorem. *The expected value of* X *is* $(1+\frac{1}{n})^{n-1}$.

Proof. Let's start by calculating the probability that $X > k$. Note that the number of ways of picking $x_1, x_2, \ldots, x_k$ from $\{1, 2, \ldots, n\}$ is n^k. Using the stars and bars method [**2**, p. 38], we can find that the number of ways such that $x_1 + \cdots + x_k = m$ for $m < n$ with $1 \le x_i \le n$ for all i is $\binom{m-1}{k-1}$. Therefore the probability that $X > k$ is

$$\sum_{m=k}^{n-1} \frac{1}{n^k}\binom{m-1}{k-1} = \frac{1}{n^k}\sum_{\ell=k-1}^{n-2}\binom{\ell}{k-1} = \frac{1}{n^k}\binom{n-1}{k}.$$

The last equality is due to the Christmas stockings theorem that states $\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1}$.

Therefore, the expected value of X is

$$\sum_{k=0}^{\infty} k \cdot \mathbb{P}(X = k) = \sum_{k=0}^{\infty}\mathbb{P}(X > k) = \sum_{k=0}^{n-1}\frac{1}{n^k}\binom{n-1}{k} = \left(1+\frac{1}{n}\right)^{n-1}.$$

∎

Remarks: Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, then as $n \to \infty$, $\mathbb{E}[X] \to e$.

Our result appears with a different proof in the unpublished online book [**1**].

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