Homework 7 Solutions
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1 Chapter 10

Problem 1. (Exercise 1)
For each of the following groups $G$, determine whether $H$ is a normal subgroup of $G$. If $H$ is a normal subgroup, write out a Cayley table for the factor group $G/H$.

(a) $G = S_4$ and $H = A_4$
(b) $G = A_5$ and $H = \{(1), (123), (132)\}$
(c) $G = S_4$ and $H = D_4$
(d) $G = Q_8$ and $H = \{1, -1, i, -i, j, -j, k, -k\}$
(e) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

Solution 1.

(a) Since $[S_4 : A_4] = 2$, then the two cosets are $A_4$ and $B_4$ (set of odd permutations). Then if $\sigma \in A_4$, $\sigma A_4 = A_4 = A_4 \sigma$. If $\sigma \notin A_4$, then $\sigma A_4 \neq A_4$, so $\sigma A_4 = B_4$. Similarly $A_4 \sigma \neq A_4$, so $A_4 \sigma = B_4$.

Therefore $\sigma A_4 = A_4 \sigma$. Therefore $A_4$ is a normal subgroup of $S_4$. The Cayley table is pretty simple:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>$A_4$</th>
<th>$B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_4$</td>
<td>$A_4$</td>
<td>$B_4$</td>
</tr>
<tr>
<td>$B_4$</td>
<td>$B_4$</td>
<td>$A_4$</td>
</tr>
</tbody>
</table>

(b) To turn in.

(c) To turn in.

(d) As a reminder

$Q_8 = \{1, -1, i, j, k, -i, -j, -k\}$

with $i^2 = j^2 = k^2 = ijk = -1$. We will show $H \leq Q_8$ by showing that if $g \in Q_8$ and $h \in H$, then $ghg^{-1} \in H$. First note that if $g \in H$, then since $H$ is a subgroup, then $ghg^{-1} \in H$. So we can assume $g \notin H$. We also know $g1g^{-1} = 1$ and $g(-1)g^{-1} = -1$. It’s also easy to see that $g(-i)g^{-1} = -g$. So we need only show that if $g \notin H$ and $h = i$, then $ghg^{-1} \in H$. We have four cases: (we will use that $ij = k$ since $ijk = -1$ and $k^2 = -1$, $jk = i$ since $ijk = -1$ and $i^2 = -1$. Also the inverse of $g$ is $-g$ for any $1 \neq g \in Q_8$)

1. $g = j$. Then $g1g^{-1} = jjj^{-1} = jij = -(jj) = -jk = -i \in H$.
2. $g = -j$. Then $g1g^{-1} = (-j)i(-j)^{-1} = -(j)i(j) = -(jj) = -jk = -i \in H$.
3. $g = k$. Then $g1g^{-1} = kik^{-1} = ki(k) = -(kik) = -kj = i \in H$.
4. $g = -k$. Then $g1g^{-1} = (-k)i(-k)^{-1} = -(k)i(k) = -(kik) = -kj = i \in H$.  

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Therefore $H \trianglelefteq Q_8$.

One could also prove it as in part a, by using that the index of $H$ in $Q_8$ is 2.

Now the Cayley table of $G/H$:

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>${j, -j, k, -k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$H$</td>
<td>${j, -j, k, -k}$</td>
</tr>
<tr>
<td>${j, -j, k, -k}$</td>
<td>${j, -j, k, -k}$</td>
<td>$H$</td>
</tr>
</tbody>
</table>

(e) To turn in.

**Problem 2. (Exercise 2)**

Find all the subgroups of $D_4$. Which subgroups are normal? What are all the factor groups of $D_4$ up to isomorphism?

**Solution 2.** To turn in.

**Problem 3. (Exercise 5)**

Show that the intersection of two normal subgroups is a normal subgroup.

**Solution 3.** Let $N_1$ and $N_2$ be normal subgroups of $G$. Let $N = N_1 \cap N_2$. Let’s show that $N \trianglelefteq G$. We will prove it by showing that $gng^{-1} \in N$ for any $g \in G$ and $n \in N$. Indeed, let $g \in G$ and $n \in N$. Since $n \in N_1 \cap N_2$, then $n \in N_1$. Since $N_1$ is normal, then $gng^{-1} \in N_1$. Therefore $gng^{-1} \in N_1$. But also $n \in N_2$ and $N_2$ is normal, so $gng^{-1} \in N_2$. Therefore $gng^{-1} \in N_1$ and $gng^{-1} \in N_2$, so $gng^{-1} \in N_1 \cap N_2 = N$. So $N \trianglelefteq G$.

**Problem 4. (Exercise 9)**

Prove or disprove: If $H$ and $G/H$ are cyclic, then $G$ is cyclic.

**Solution 4.** To turn in.

**Problem 5. (Exercise 10)**

Let $H$ be a subgroup of index 2 of a group $G$. Prove that $H$ must be a normal subgroup of $G$. Conclude that $S_n$ is not simple for $n \geq 3$.

**Solution 5.** To turn in.

**Problem 6. (Exercise 11)**

If a group $G$ has exactly one subgroup $H$ of order $k$, prove that $H$ is normal in $G$.

**Solution 6.** To turn in.

**Problem 7. (Exercise 13)**

Recall that the **center** of a group $G$ is the set

$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G \}$.

(a) Calculate the center of $S_3$.

(b) Calculate the center of $GL_2(\mathbb{R})$.

(c) Show that the center of any group $G$ is a normal subgroup of $G$.

(d) If $G/Z(G)$ is cyclic, show that $G$ is abelian.

**Solution 7.**

(a) To turn in.
(b) We want to find all invertible matrices \(M\) that satisfy \(AM = MA\) for any invertible matrix \(A\). Let

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Now since \(M\) commutes with all elements of \(GL_2(\mathbb{R})\), then in particular, it commutes with \(J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

Therefore

\[
JM = \begin{pmatrix} c & d \\ a & b \end{pmatrix} = MJ = \begin{pmatrix} b & a \\ d & c \end{pmatrix}.
\]

Therefore \(a = d\) and \(b = d\). Therefore \(M\) is of the form

\[
M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.
\]

Now, \(M\) should also commute with \(K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\), so

\[
KM = \begin{pmatrix} (a + b) & (a + b) \\ a & b \end{pmatrix} = MK = \begin{pmatrix} (a + b) & a \\ (a + b) & b \end{pmatrix}.
\]

Therefore \(a + b = a\), so \(b = 0\). Therefore

\[
M = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.
\]

And indeed if

\[
\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in GL_2(\mathbb{R}),
\]

then

\[
\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & aw \end{pmatrix},
\]

and

\[
\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & aw \end{pmatrix}.
\]

So \(M\) commutes with every element of \(GL_2(\mathbb{R})\), so \(M\) is in the center. Yet every element in the center must be of this form. Therefore

\[
Z(GL_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0 \right\}
\]

(c) To turn in.

(d) Suppose that \(G/Z(G)\) is cyclic. Then \(G/Z(G) = \langle gZ(G) \rangle\) for some \(g \in G\). Let \(g_1, g_2 \in G\). We want to show that \(g_1g_2 = g_2g_1\) (to show \(G\) is abelian). Since \(G/Z(G) = \langle gZ(G) \rangle\), then \(g_1Z(G) = g^{k_1}Z(G)\) for some integer \(k_1\) and \(g_2Z(G) = g^{k_2}Z(G)\) for some integer \(k_2\). Then \(g_1 \in g^{k_1}Z(G)\), so there exists an \(x \in Z(G)\) such that \(g_1 = g^{k_1}x\). Similarly, there exists a \(y \in Z(G)\) such that \(g_2 = g^{k_2}y\). Therefore

\[
g_1g_2 = (g^{k_1}x)(g^{k_2}y) = (xyg^{k_1})(g^{k_2}y) = x(g^{k_1+k_2})y = xyg^{k_1+k_2}.
\]
We used that $x$ and $y$ commute with any element of $G$. Similarly:

$$g_2g_1 = (g^{k_2} y)(g^{k_1} x)$$
$$= (yg^{k_2})(g^{k_1}x)$$
$$= y(g^{k_2+k_1})x$$
$$= x g^{k_1+k_2}$$

Therefore $g_1g_2 = g_2g_1$. 
