On the maximum number of consecutive integers on which a character is constant

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Preliminaries

1. Let $\chi$ be a non-principal Dirichlet character to the prime modulus $p$.
2. Let $H(p)$ be the maximum number of consecutive integers for which $\chi$ is constant.
3. Trivially $H(p) \leq p$.
4. By the Pólya–Vinogradov inequality, $H(p) \ll p^{1/2} \log p$.
5. By the Burgess inequality, $H(p) \ll_{\varepsilon} p^{1/4+\varepsilon}$.
Explicit Estimates

Let $\chi$ be a non-principal Dirichlet character to the prime modulus $p$. Let $H(p)$ be the maximum number of consecutive integers for which $\chi$ is constant.

Theorem (Burgess, 1963)

$$H(p) = O(p^{1/4} \log p).$$

Theorem (McGown, 2011)

$$H(p) < \left\{ \frac{\pi e\sqrt{6}}{3} + o(1) \right\} p^{1/4} \log p.

Furthermore,

$$H(p) \leq \begin{cases} 
7.06p^{1/4} \log p, & \text{for } p \geq 5 \cdot 10^{18}, \\
7p^{1/4} \log p, & \text{for } p \geq 5 \cdot 10^{55}.
\end{cases}$$

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In a conference in 1973, Norton made the following claim without proof:

**Claim**

\[
H(p) \leq 2.5 p^{1/4} \log p \text{ for } p > e^{15} \approx 3.27 \times 10^6 \text{ and } \\
H(p) < 4.1 p^{1/4} \log p \text{ for all odd } p.
\]

I was able to prove the claim (and a little more), improving on the theorem of McGown.
Consecutive integers where a character is constant

Main Theorem

**Theorem (T)**

Let $\chi$ be a non-principal Dirichlet character to the prime modulus $p$. Let $H(p)$ be the maximum number of consecutive integers for which $\chi$ is constant, then

$$H(p) < \left\{ \frac{\pi}{2} \sqrt{\frac{e}{3}} + o(1) \right\} p^{1/4} \log p.$$ 

Furthermore,

$$H(p) \leq \begin{cases} 3.64 p^{1/4} \log p, & \text{for all odd } p, \\ 1.55 p^{1/4} \log p, & \text{for } p \geq 2.5 \cdot 10^9. \end{cases}$$
The main ingredient in the proof comes from estimating

\[ S_{\chi}(h, w) = \sum_{m=1}^{p} \left| \sum_{l=0}^{h-1} \chi(m + l) \right|^2. \]

- Burgess showed that
  \[ S_{\chi}(h, w) < (4w)^{w+1} ph^w + 2wp^{1/2} h^{2w}. \]

- McGown improved it to
  \[ S_{\chi}(h, w) < \frac{1}{4} (4w)^w ph^w + (2w - 1)p^{1/2} h^{2w}. \]

- For quadratic characters, Booker showed
  \[ S_{\chi}(h, w) < \frac{(2w)!}{2^w w!} ph^w + (2w - 1)p^{1/2} h^{2w}. \]

- I showed that Booker’s inequality holds for all characters.
Lower bound Lemma

Let $h$ and $w$ be positive integers. Let $\chi$ be a non-principal Dirichlet character to the prime modulus $p$ which is constant on $(N, N + H]$ and such that

$$4h \leq H \leq \left(\frac{h}{2}\right)^{2/3} p^{1/3}.$$

Let $X := H/h$, then $X \geq 4$ and

$$S_\chi(h, w) \geq \left(\frac{3}{\pi^2}\right) X^2 h^{2w+1} g(X) = AH^2 h^{2w-1} g(X),$$

where $A = \frac{3}{\pi^2}$, and

$$g(X) = 1 - \left(\frac{13}{12AX} + \frac{1}{4AX^2}\right).$$
Let $a$ and $b$ be integers satisfying $1 \leq a \leq \left\lfloor \frac{2H}{h} \right\rfloor$ and

$$\left| a \frac{N}{p} - b \right| \leq \frac{1}{\left\lfloor \frac{2H}{h} \right\rfloor + 1} \leq \frac{h}{2H}.$$ 

Now define $I(q, t)$ to be the real interval:

$$I(q, t) := \left( \frac{N + pt}{q}, \frac{N + H + pt}{q} \right),$$

for integers $0 \leq t < q \leq X$ and $\gcd(at + b, q) = 1$. 

Enrique Treviño

JMM 2012
Given

\[ l(q, t) := \left( \frac{N + pt}{q}, \frac{N + H + pt}{q} \right), \]

we have

- \( \chi \) is constant inside the interval \( l(q, t) \) since if \( m \in l(q, t) \), then

\[ \chi(q)\chi(m) = \chi(qm) = \chi(qm - pt) = \chi(N + i), \]

where \( i \in (0, H] \).

- The \( l(q, t) \) are disjoint (for this you need to use the restriction on \( H \) and on \( a \)).

- \( l(q, t) \subset (0, p) \).
Since the \( I(q, t) \) are disjoint and they are contained in \((0, p)\), we have

\[
S_{\chi}(h, w) = \sum_{m=0}^{p-1} \left| \sum_{l=0}^{h-1} \chi(m + l) \right|^{2w} \geq \sum_{q,t} \sum_{m \in I(q,t)} \left| \sum_{l=0}^{h-1} \chi(m + l) \right|^{2w} \\
\geq h^{2w} \sum_{q,t} \left( \frac{H}{q} - h \right) = h^{2w+1} \sum_{q \leq X} \sum_{0 \leq t < q} \left( \frac{X}{q} - 1 \right).
\]

Recall that \( X = H/h \). Evaluating this last sum yields our lemma. The main difference between the technique Burgess and McGown use is that they have \( X = H/(2h) \) and then they evaluate the last sum with 1 instead of \( \left( \frac{X}{q} - 1 \right) \).
Combining the upper and lower bounds on $S_{\chi}(h, w)$ we have

$$AH^2 h^{2w-1} g(X) \leq S_{\chi}(h, w) < \frac{(2w)!}{2^w w!} ph^w + (2w - 1)p^{1/2} h^{2w}.$$ 

Optimizing for $h$ and $w$ asymptotically, we find the asymptotic in our theorem and we also prove that $H(p) < 1.55p^{1/4} \log p$ for $p \geq 10^{64}$.

Since $H < \left(\frac{h}{2}\right)^{2/3} p^{1/3}$, we have to juggle a bit and we find that we are constrained to $p \geq 2.5 \times 10^9$.

To cover the gaps between $2.5 \times 10^9$ and $10^{64}$ we pick specific $h$’s and $w$’s and check for intervals as depicted in the table on the following slide.
Table: As an example on how to read the table: when $w = 10$ and $h = 62$, then the constant $1.55$ works for all $p \in [10^{18}, 10^{19}]$. It is also worth noting that the inequality $1.55p^{1/4} \log p < h^{2/3}p^{1/3}$ is also verified for each choice of $w$ and $h$. 
For all $p$

To get a bound for all $p$ we use the following theorem (established with elementary methods):

**Theorem (Brauer)**

\[ H(p) < \sqrt{2p} + 2. \]

Using this, one can show that $H(p) < 3.64p^{1/4} \log p$ whenever $p < 3 \times 10^6$. Using the techniques from before one can show that for $p \geq 3 \times 10^6$, $H(p) < 3.64p^{1/4} \log p$. One of the obstacles preventing us from getting a lower number is the restriction $H < \left( \frac{h}{2} \right)^{2/3} p^{1/3}$. 
Thank you!