Cardinality Homework Solutions

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Problem 1. In the following problems, find a bijection from \( A \) to \( B \) (you need not prove that the function you list is a bijection):

(a) \( A = (-3, 3), \ B = (7, 12) \).

(b) \( A = (0, 2), \ B = (0, 1) \).

(c) \( A = (1, 7), \ B = (-2, 2) \).

(d) \( A = \mathbb{N}, \ B = \mathbb{Z} \).

(e) \( A = \mathbb{R}, \ B = (0, \infty) \).

(f) \( A = \mathbb{N}, \ B = \{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\} \).

(g) \( A = \{0, 1\} \times \mathbb{N}, \ B = \mathbb{N} \).

(h) \( A = [0, 1], \ B = (0, 1) \).

Solution 1. (a) \( f(x) = \frac{5}{6}x + \frac{57}{6} \).

(b) \( f(x) = \frac{1}{2^x} \).

(c) \( f(x) = \frac{2}{3}x - \frac{8}{3} \).

(d) The idea is to have \( f(1) = 0 \) then alternate \( f(2) = 1, \ f(3) = -1, \ f(4) = 2, \ f(5) = -2 \) and so on. In a pretty math formula it looks as follows:

\[
f(n) = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even,} \\
-\frac{n-1}{2}, & \text{if } n \text{ is odd.}
\end{cases}
\]

(e) \( f(x) = e^x \).

(f) \( f(x) = \frac{\sqrt{2}}{n} \).

(g) Let \((a,b) \in \{0,1\} \times \mathbb{N}\). The idea is to send the terms with \( a = 0 \) to the even integers and the \( a = 1 \) terms to the odd positive integers. In a pretty formula it looks as follows:

\[
f((a,b)) = \begin{cases} 
2b, & \text{if } a = 0, \\
2b - 1, & \text{if } a = 1.
\end{cases}
\]
(h) We discussed in class that the strategy for this was to find a countable subset and play around with it (in this case “deleting” two terms from it). A possible bijection is the following:

\[
f(x) = \begin{cases} 
\frac{1}{2}, & \text{if } x = 0, \\
\frac{1}{n+2}, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\
x, & \text{otherwise.}
\end{cases}
\]

**Problem 2.** Prove or disprove that the following sets are countable:

(a) \(A = \{\log n : n \in \mathbb{N}\}\).

(b) \(A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \leq n\}\).

(c) \(A = \mathbb{Q}^{100}\).

(d) The set of irrational numbers.

**Solution 2.**

(a) Let \(f : \mathbb{N} \to A\) be defined as \(f(n) = \log n\). \(f\) is one-to-one because \(\log x = \log y\) implies \(x = y\). \(f\) is onto because the image of \(f\) is exactly the set \(A\). Therefore \(f\) is a bijection. Therefore \(A\) is countable.

(b) \(A \subseteq \mathbb{N} \times \mathbb{N}\). Therefore \(|A| \leq |\mathbb{N}|\). But since \(A\) is also infinite (indeed, \(n\) is unbounded in the set), then \(|A| = \mathbb{N}\). Therefore \(A\) is countable.

(c) \(\mathbb{Q}^{100} = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}\) is a cross product of countable sets. Since the finite cross product of countable sets is countable. The set is countable.

(d) Let \(A\) be the set of irrationals. Since every real number is either rational or irrational, then \(A \cup \mathbb{Q} = \mathbb{R}\). Suppose for the sake of contradiction that \(A\) is countable. Then since \(\mathbb{Q}\) is countable, \(A \cup \mathbb{Q}\) is also countable. Therefore \(\mathbb{R}\) is countable. This contradicts the fact that the set of real numbers is uncountable. Therefore \(A\) is uncountable.

**Problem 3.** Let \(A\) and \(B\) be sets. Prove that if \(|A| \leq |B|\) and \(|B| \leq |A|\), then \(|A| = |B|\).

**Remark 1.** This result is known as the Cantor-Bernstein-Schröder Theorem.

**Solution 3.** Because \(|A| \leq |B|\), there exists a one-to-one function \(f : A \to B\). Because \(|B| \leq |A|\), there exists a one-to-one function \(g : B \to A\) and hence there is a bijection \(g^{-1} : \text{Im}(g) \to B\). Note that \(g^{-1}\) is already a bijection, so consider the following function:

\[
h_1(a) = \begin{cases} 
g^{-1}(a), & \text{if } a \in \text{Im}(g), \\
f(a), & \text{if } a \in A - \text{Im}(g).
\end{cases}
\]

Because \(g^{-1}\) is onto, \(h_1\) is onto, but it’s not one-to-one because any element in the image of \(f\) is also in the image of \(g^{-1}\). So we have an onto function but it’s not one-to-one yet. It does however only fail one-to-one-ness in a subset of the image of \(f\). We can try to shrink the parts where it fails one-to-one-ness. Consider the following function:

\[
h_2(a) = \begin{cases} 
g^{-1}(a), & \text{if } a \in (\text{Im}(g) - \text{Im}(g \circ f)) \cup (\text{Im}(g \circ f \circ g)), \\
f(a), & \text{if } a \in (A - \text{Im}(g)) \cup (\text{Im}(g \circ f) - \text{Im}(g \circ f \circ g)).
\end{cases}
\]

\(h_2\) is onto by construction (using that \(g^{-1}\) is a bijection) and now the space where \(h_2\) fails to be one-to-one is in a subset of the image of \(f \circ g \circ f\) which is smaller than the image of \(f\). We can keep iterating this process and shrink the area that fails the one-to-one-ness while keeping the onto-ness. The question is whether this shrinking leads to the empty set eventually. Note that in \(h_1\) we use \(f(a)\) whenever \(a \in A - \text{Im}(g)\) and in \(h_2\)
we use $f(a)$ whenever $a \in (A - \text{Im}(g)) \cup (\text{Im}(g \circ f) - \text{Im}(g \circ f \circ g)) = (A - \text{Im}(g)) \cup ((g \circ f)(A - \text{Im}(g)))$.

There’s a pattern here that can be extended. Let

$$
G_0 = A - \text{Im}(g), \\
G_1 = \{g \circ f(a) \mid a \in G_0\}, \\
G_n = \{g \circ f(a) \mid a \in G_{n-1}\} \text{ for } n \geq 2.
$$

Now let

$$
G = \bigcup_{i=0}^{\infty} G_i.
$$

Since every $G_i$ is a subset of $A$, then $G \subseteq A$. Now define the following function:

$$
h(a) = \begin{cases} 
  f(a), & \text{if } a \in G, \\
  g^{-1}(a), & \text{if } a \notin G.
\end{cases}
$$

$h$ is a function from $A$ to $B$ because $h$ has a unique output for each input $a \in A$ and because the image of $h$ is a subset of $B$ (since $f$ and $g^{-1}$ are functions to $B$).

Now let’s prove that $h$ is a bijection. First we’ll show that $h$ is one-to-one. Suppose $h(x) = h(y)$ for $x, y \in A$. We have three cases:

(i) If $x$ and $y$ are in $G$, then $h(x) = h(y)$ implies $f(x) = f(y)$ and since $f$ is one-to-one it implies that $x = y$.

(ii) If $x$ and $y$ are not in $G$, then $h(x) = h(y)$ implies $g^{-1}(x) = g^{-1}(y)$ and since $g^{-1}$ is one-to-one then $x = y$.

(iii) If $x$ is in $G$ and $y$ is not in $G$, then $h(x) = h(y)$ implies $f(x) = g^{-1}(y)$, therefore $g \circ f(x) = y \notin G$. But since $x \in G$ that means there exists an integer $k$ such that $x \in G_k$. Since $x \in G_k$, then $g \circ f(x) \in G_{k+1}$, so $g \circ f(x) \in G$. This contradicts that $y \notin G$.

Note that we don’t need the fourth case $x \notin G$, $y \in G$ because it is analogous to the process in case (iii). Since $h(x) = h(y)$ implies $x = y$, then $h$ is one-to-one.

To finish the theorem we need only prove $h$ is onto. Let $y \in B$. We’ll prove it by considering two cases.

(i) If $g(y) \notin G$, then $h(g(y)) = g^{-1}(g(y)) = y$, therefore $y \in \text{Im}(h)$.

(ii) If $g(y) \in G$, then there exists an integer $k$ such that $g(y) \in G_k$. Since $g(y) \notin A - \text{Im}(g)$ (because $g(y)$ must be inside the image of $g$), then $k \geq 1$. Since $k \geq 1$ and $g(y) \in G_k$, then $g(y) = g \circ f(x)$ for some element $x \in G_{k-1}$. But then $x \in G$, so $h(x) = f(x) = g^{-1}(g(y)) = y$ and hence $y \in \text{Im}(h)$.

Therefore $h$ is onto and hence we have a bijection from $A$ to $B$ proving that $|A| = |B|$.

**Problem 4.** Prove that $|(0, 1)| = |[0, 1]|$.

**Solution 4.** We can prove it by showing that the function in exercise (g) of Problem 1 is a bijection but we can prove it quickly by using the Cantor-Bernstein-Schöder theorem as follows:

(1) $|(0, 1)| \leq |[0, 1]|$ because $(0, 1) \subseteq [0, 1]$. Since $(0, 1) = \mathbb{R}$ this means $|\mathbb{R}| \leq |(0, 1)|$.

(2) $|[0, 1]| \leq |\mathbb{R}|$ because $[0, 1] \subseteq \mathbb{R}$.

Combining (1) and (2) using the Cantor-Bernstein-Schöder theorem we conclude that $|(0, 1)| = |\mathbb{R}| = |(0, 1)|$.

**Problem 5.** Dedekind decided he wanted to write a definition of an infinite set that did not depend on the natural numbers. He defined it as follows: “A is an infinite set if there exists a proper subset $B$ of $A$ (that is, $B \subseteq A$ and $A \neq B$) such that $|A| = |B|$.” We’ll call sets satisfying this condition “Dedekind-infinite” sets.
(a) Prove that if A is a finite set, then A is not Dedekind-infinite.

(b) Prove that if A is an infinite set, then A is Dedekind-infinite.

Note that proving (a) and (b) means that the natural definition of an infinite set (saying that it is not finite) is equivalent to the Dedekind definition.

**Solution 5.** For part (a) suppose for the sake of contradiction that A is finite and A is Dedekind infinite, i.e., there exists a proper subset B of A such that |A| = |B|. Since A is finite let |A| = |B| = n for some integer n. Since B is a proper subset of A, then there exists an element a ∈ A – B. A – B and B are disjoint and infinite, therefore

\[ n = |A| = |(A - B) \cup B| = |A - B| + |B| = 1 + n. \]

n ≥ n + 1 is a contradiction, therefore A is not Dedekind infinite.

Part b is harder. Let A be an infinite set. The goal is to find a proper subset B of A such that |A| = |B|.

The idea is to let B be A minus one element. Let’s figure out how to take one element of A without losing cardinality. Since A is infinite, then we can find a countable subset of A as follows:

- Since A is non-empty, there exists an a₁ ∈ A. Let A₁ = A - {a₁}, since A is infinite, A₁ is also infinite.
- Since A₁ is non-empty, there exists an a₂ ∈ A₁. Let A₂ = A₁ - {a₂}. Since A is infinite, A₂ is also infinite.
- Let n ≥ 2. Since Aₙ₋₁ is non-empty, there exists an aₙ ∈ Aₙ₋₁. Let Aₙ = A - {a₁, a₂, . . . , aₙ₋₁, aₙ}. Since A is infinite and {a₁, . . . , aₙ₋₁, aₙ} is finite, then Aₙ is infinite.

Following that process we can find an infinite subset \( \{ a₁, a₂, a₃, . . . \} \) of A. Now define the following function \( f : A \to A₁ \):

\[
f(x) = \begin{cases} 
a_{n+1}, & \text{if } x = aₙ \text{ for some } n \in \mathbb{N}, 
x, & \text{otherwise.}
\end{cases}
\]

The function is one-to-one because if \( f(x) = f(y) \) then one of three cases occurs:

- If \( x = aₙ \) and \( y = a_m \) for some \( n \) and \( m \), then \( f(x) = f(y) \) implies \( a_{n+1} = a_{m+1} \), so \( n = m \), so \( aₙ = a_m \), and hence \( x = y \).
- If \( x = aₙ \) for some \( n \) and \( y \neq a_m \) for any \( m \), then \( f(x) = f(y) \) implies \( a_{n+1} = y \), but that contradicts that \( y \) is not of the form \( a_m \).
- If \( x \neq aₙ \) for any \( n \) and \( y \neq a_m \) for any \( m \), then \( f(x) = f(y) \) implies \( x = y \).

Therefore \( f \) is one-to-one. \( f \) is onto because if \( y \in A₁ \), then either \( y = aₙ \) for some integer \( n \geq 2 \) and hence \( f(aₙ₋₁) = aₙ = y \) or \( y \neq aₙ \) for any integer \( n \geq 2 \) and then \( f(y) = y \) unless \( y = a₁ \). Since \( y \in A₁ \), then \( y \neq a₁ \). Therefore for any \( y \in A₁ \), there exists an \( x \in A \) such that \( f(x) = y \). This proves \( f \) is onto. Since \( f \) is onto and one-to-one, \( |A| = |A₁| \). Since \( A₁ \subseteq A \) and \( A₁ \neq A \), then A is Dedekind infinite.

**Problem 6.** Let \( \mathcal{F} \) be the set of all functions \( \mathbb{N} \to \{0, 1\} \). Show that \( |\mathbb{R}| = |\mathcal{F}| \).

**Solution 6.** We’ll prove that \( |\mathcal{F}| = |2^\mathbb{N}| \) and since \( |2^\mathbb{N}| = |\mathbb{R}| \), then \( |\mathcal{F}| = |\mathbb{R}| \). To prove that \( |\mathcal{F}| = |2^\mathbb{N}| \), we’ll build a bijection \( h : \mathcal{F} \to 2^\mathbb{N} \) as follows. For a function \( f \in \mathcal{F} \), i.e., \( f : \mathbb{N} \to \{0, 1\} \), define \( h(\mathcal{F}) \) to be \( \{ n \in \mathbb{N} \mid f(n) = 1 \} \) i.e., the set of natural numbers that satisfy that \( f(n) = 1 \).

First let’s prove that \( h \) is onto. Suppose \( A \subseteq 2^\mathbb{N} \), then \( A \subseteq \mathcal{F} \). Let \( f \) be defined as follows:

\[
f(n) = \begin{cases} 
1, & \text{if } n \in A, 
0, & \text{if } n \notin A.
\end{cases}
\]

Then \( h(f) = \{ n \in \mathbb{N} \mid f(n) = 1 \} = A \). Therefore \( h \) is onto.

Now let’s prove that \( h \) is one-to-one. Suppose \( h(f) = h(g) \). Let’s prove that \( f = g \). Since \( f : N \to \{0, 1\} \) and \( g : N \to \{0, 1\} \), then \( \text{dom}(f) = \text{dom}(g) = \mathbb{N} \), so to show \( f = g \) we need only show \( f(n) = g(n) \) for all \( n \in \mathbb{N} \). Note that there are two cases:
• If $f(n) = 1$, then $n \in h(f)$, but since $h(f) = h(g)$, then $n \in h(g)$, therefore $g(n) = 1$. Therefore $f(n) = g(n) = 1$.

• If $f(n) = 0$, then $n \not\in h(f)$, but since $h(f) = g(f)$, then $n \not\in h(g)$, therefore $g(n) = 0$. Therefore $f(n) = g(n) = 0$.

In both cases $f(n) = g(n)$, which allows us to conclude that $f = g$ and hence $h$ is one-to-one.

Problem 7. Let $\mathcal{F}$ be the set of all functions $\mathbb{R} \to \{0, 1\}$. Show that $|\mathbb{R}| < |\mathcal{F}|$.

Solution 7. Let $h : \mathcal{F} \to 2^{\mathbb{R}}$ be defined by $h(f) = \{x \in \mathbb{R} \mid f(x) = 1\}$. Just like in the previous problem, this function is a bijection. Therefore $|\mathcal{F}| = |2^{\mathbb{R}}|$. By Cantor’s theorem we know $|\mathbb{R}| < |2^{\mathbb{R}}| = |\mathcal{F}|$. Which is what we wanted to prove.