Report on the 65th Annual International Mathematical Olympiad

Béla Bajnok, John Berman & Enrique Treviño

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BÉLA BAJNOK
Gettysburg College
Gettysburg, PA 17325
bbajnok@gettysburg.edu

JOHN BERMAN
Raleigh, NC
mathbyberman@gmail.com

ENRIQUE TREVIÑO
Lake Forest College
Lake Forest, IL 60045
trevino@lakeforest.edu

The International Mathematical Olympiad (IMO) is the world's leading mathematics competition for high school students and is organized annually by different host countries. The competition consists of three problems each on two consecutive days, with an allowed time of four and a half hours both days. In recent years, more than one hundred countries have sent teams of up to six students to compete.

The 65th IMO was organized by the United Kingdom, and it was held in Bath between July 11 and July 22, 2024, with the participation of 609 contestants from 108 countries.

Each year, the members of the US team are chosen during the Math Olympiad Program (MOP), a year-long endeavor organized by the MAA's American Mathematics Competitions (AMC) program. Students gain admittance to MOP based on their performance on a series of examinations, culminating in the USA Mathematical Olympiad (USAMO). A report on the 2024 USAMO can be found in the February 2025 issue of this *Magazine*; a similar report on the 2024 USA Junior Mathematical Olympiad appeared in the January 2025 issue of the *College Mathematics Journal*. More information on the American Mathematics Competitions program can be found on the site https://maa.org/student-programs/amc/.

The members of the 2024 US team were Jordan Lefkowitz (Grade 12, E.O. Smith High School, CT); Krishna Pothapragada (Grade 12, Naperville North High School, IL); Linus Tang (Grade 12, Davidson Academy Online, CA); Jessica Wan (Grade 12, Homeschooled, FL); Alexander Wang (Grade 10, Millburn High School, NJ); and Qiao (Tiger) Zhang (Grade 11, Sierra Canyon School, CA). Pothapragada, Tang, Wan, Wang, and Zhang each earned Gold Medals; and Lefkowitz received a Silver Medal. In the unofficial ranking of countries, the United States finished first, ahead of China (second) and South Korea (third).

Below we present the problems and solutions of the 65th IMO. Our solutions are those of the current authors, utilizing some of the various sources already available. Each problem was worth 7 points; the nine-tuple ($a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0$; $\mathbf{a}$) states the number of students who scored $7, 6, \ldots, 0$ points, respectively, followed by the mean score achieved for the problem.

Problem 1 (413, 31, 12, 13, 35, 22, 54, 29; **5.570**); *proposed by Colombia.* Determine all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \lfloor 3\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is divisible by n.

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Solution. Let us say that a real number α is *good* when it satisfies the requirement that for every positive integer n,

$$S_n(\alpha) = \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \lfloor 3\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is divisible by n. We will prove that α is good if and only if it is an even integer.

First, note that

$$S_n(\alpha + 2) = S_n(\alpha) + (2 + 4 + \cdots + 2n) = S_n(\alpha) + n(n+1),$$

so α is good if and only if $\alpha + 2$ is good. Therefore we can restrict our attention to an interval of length 2; in particular, it suffices to show that the only good real number α with $-1 < \alpha \le 1$ is $\alpha = 0$.

Since $S_n(0) = 0$ is divisible by n for every n, $\alpha = 0$ is clearly good. It is also easy to see that $\alpha = 1$ is not good, as $S_n(1) = n(n+1)/2$, which is not divisible by n when n is even.

To verify that no value of α with $0 < \alpha < 1$ is good, note that if n is the smallest positive integer for which $n\alpha \ge 1$, then $n \ge 2$ and $n\alpha = (n-1)\alpha + \alpha < 1 + 1 = 2$, so

$$S_n(\alpha) = \underbrace{0 + 0 + \cdots + 0}_{n-1 \text{ terms}} + 1 = 1,$$

which is not divisible by n. Similarly, when $-1 < \alpha < 0$ and n is the smallest positive integer for which $n\alpha < -1$, then $n \ge 2$ and $n\alpha = (n-1)\alpha + \alpha \ge -1 - 1 = -2$, so

$$S_n(\alpha) = \underbrace{-1 - 1 - \cdots - 1}_{n-1 \text{ terms}} - 2 = -n - 1,$$

which is not divisible by n. Our proof is thus complete.

Problem 2 (156, 6, 9, 2, 10, 129, 80, 217; **2.544**); *proposed by Indonesia.* Determine all pairs (a, b) of positive integers for which there exist positive integers g and N such that

$$\gcd(a^n + b, b^n + a) = g$$

holds for all integers $n \ge N$.

Solution. For positive integers a, b, and n, we set $x_n = \gcd(a^n + b, b^n + a)$. Since for $a = b = 1$ we have $x_n = 2$ for every positive integer n, we see that $(a, b) = (1, 1)$ satisfies the requirement of the problem. We will prove that there are no other pairs.

Suppose that there exist positive integers g and N such that

$$\gcd(a^n + b, b^n + a) = g$$

holds for all integers $n \ge N$. Let $c = ab + 1$, and let $M > N$ be any multiple of $\phi(c)$ (the number of positive integers up to c that are relatively prime to c).

Recall that, by Euler's theorem, $k^{\phi(c)} \equiv 1 \bmod c$ whenever k is relatively prime to c. Since a and b are both relatively prime to c, we then have $a^M \equiv 1$ and $b^M \equiv 1 \bmod c$. Therefore, $a^M + c - 1 = a(a^{M-1} + b)$ and $b^M + c - 1 = b(b^{M-1} + a)$ are divisible by c, and thus $a^{M-1} + b$ and $b^{M-1} + a$ are as well. This then means that x_{M-1} must be divisible by c, and since $M - 1 \ge N$, we can conclude that g is divisible by c. In particular, $x_M = \gcd(a^M + b, b^M + a)$ and $x_{M+1} = \gcd(a^{M+1} + b, b^{M+1} + a)$ are both divisible by c, which implies that $a^M + b$, $b^M + a$, $a^{M+1} + b$, and $b^{M+1} + a$ are all divisible by c. Then, since $a^M \equiv 1$ and $b^M \equiv 1 \bmod c$, we must have that $b+1$, $a+1$, and $a+b$ are divisible by c. Therefore, we get that $(b+1)+(a+1)-(a+b)=2$ is divisible by c, which can only happen if $a=b=1$, as claimed.

Problem 3 (8, 7, 2, 3, 10, 38, 40, 501; **0.437**); *proposed by Australia.* Let $a_1, a_2, a_3, \dots$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, \dots, a_{n-1}$. Prove that at least one of the sequences $a_1, a_3, a_5, \dots$ and $a_2, a_4, a_6, \dots$ is eventually periodic.

Solution. We represent the sequence by an infinite array, with rows and columns indexed by the positive integers, so that row r lists in increasing order all positive integers n for which $a_n = r$; we also have the initial indices $1, \dots, N$ boxed (see Figure 1 for an example). The condition in the problem statement may be reformulated: An integer $n > N$ (that is, an unboxed integer) is in row i if and only if $n-1$ is in column i.

2	10	15	**20**	**26**	**32**	**38**	⋯
1	3	6	11	13	**18**	**24**	⋯
4	5	7	8	**16**	**22**	**28**	⋯
9	12	21					
14	17	27	*empty region*				
19	23	33	*(Claim 4)*				
25	29	39					
⋮	⋮	⋮					

Figure 1 Part of the array corresponding to $N = 6$ and the sequence $(2, 1, 2, 3, 3, 2, 3, 3, 4, 1, 2, \dots)$.

We carry out our proof through a sequence of claims.

Claim 1. Row r is infinite if and only if column r is infinite.

Proof of Claim 1. If row r is infinite, then $a_i = r$ holds for infinitely many indices i. Therefore, if $i > N$, then a_{i-1} appears in the sequence at least r times, so the sequence contains infinitely many positive integers at least r times, and thus column r is infinite. The other direction is similar. □

Claim 2. There is an infinite row.

Proof of Claim 2. If not, then every column must be finite by Claim 1. But any nonempty row must contain an entry in column 1, so there are only finitely many nonempty rows. Since each row is finite, this would mean that there are only finitely many entries in the array, which is a contradiction. □

Claim 3: If row r is infinite for some $r \geq 2$, then row $r-1$ is infinite.

Proof of Claim 3. By Claim 1, it suffices to prove that if column r is infinite then column $r-1$ is infinite, and this follows from the fact that, by construction, any row that contains an entry in column r must also contain a smaller entry in column $r-1$. □

Claim 4. There exists an integer K such that rows $1, \dots, K$ are infinite and all other rows are finite.

Proof of Claim 4. Let $M = \max(a_1, \ldots, a_N)$; we prove that the region of our array below row M and to the right of column M has no entries. For the sake of a contradiction, let n be the smallest number appearing in this region, and say it is in row r. Since the numbers in row r are in increasing order, n must be in column $M + 1$; we let the $M + 1$ smallest numbers in row r be $n_1 < \cdots < n_{M+1} = n$. By definition of M, nothing in row r is boxed, so $n_1 - 1 < \cdots < n_{M+1} - 1$ are all in column r. By the Pigeonhole Principle, these $M + 1$ numbers are not all in the first M rows; since these numbers are all less than n, this contradicts the minimality of n. This then means that the array has only finitely many (but, by Claim 2, at least one) infinite rows, and our claim now follows from Claim 3. □

Now let us define C to be the smallest positive integer that is greater than or equal to all of

- $M = \max(a_1, \ldots, a_N)$,
- all numbers that are in both the first K rows and first K columns,
- all numbers that are in neither the first K rows nor first K columns,
- any number in the first K rows that is in the same column as a boxed number, and
- any number in the first K columns that is in the same row as a boxed number.

We will also say that a number is *rowlike* if it is in the first K rows and greater than C and *columnlike* if it is in the first K columns and greater than C. (In Figure 1, the rowlike numbers are in bold.)

Claim 5. All rowlike numbers are the same parity, and all columnlike numbers are the opposite parity. (That is, the original sequence eventually alternates between evens and odds.)

Proof of Claim 5. Suppose that $n > C$. By the condition given in the problem statement, if n is in the first K columns, then $n + 1$ is in the first K rows. We show that if n is in the first K rows, then $n + 1$ is in the first K columns. Indeed, since n cannot be in the first K columns and is not boxed, $n + 1$ must not be in the first K rows, so $n + 1$ must be in the first K columns. Hence the numbers $n = C + 1, C + 2, C + 3, \ldots$ alternate between the first K rows and first K columns, proving Claim 5. □

Let us set $C_{n,i}$ to be the number of entries in column i that are at most n, and $R_{n,i}$ to be the number of entries in row i row that are at most n.

Claim 6. If n is rowlike and in column i, then $n + 2$ is in row $C_{n,i}$. (For example, in Figure 1, 28's column contains 2 numbers less than or equal to 28, so the next rowlike number 30 must be in row 2.)

Proof of Claim 6. If $n > C$ is in column i, then by the definition of C, neither column i nor row i contains any boxed numbers, so if $n_1 < \cdots < n_{C_{n,i}} = n$ are the numbers in column i less than or equal to n, then $n_1 + 1 < \cdots < n_{C_{n,i}} + 1 = n + 1$ are the smallest $C_{n,i}$ numbers in row i. In particular, $n + 1$ is the $C_{n,i}^{\text{th}}$ number in row i, so it is in column $C_{n,i}$. Therefore, $n + 2$ is in row $C_{n,i}$, proving Claim 6. □

Define the K-tuple $A_n = (a_n; R_{n,2} - R_{n,1}, R_{n,3} - R_{n,1}, \ldots, R_{n,K} - R_{n,1})$. The first coordinate a_n records the row of n, and the rest of the k-tuple describes the relative shape of the first K rows of the grid when only the indices $1, \ldots, n$ are filled in. (For example, in the example of Figure 1 we have $A_{22} = A_{28} = (3; 2, 2)$.)

Claim 7. If m and n are rowlike and $A_m = A_n$, then $A_{m+2} = A_{n+2}$.

Proof of Claim 7. From $A_m = A_n$ we get $R_{n,1} - R_{m,1} = \cdots = R_{n,K} - R_{m,K}$, so the column of n has an entry less than or equal to n in row j if and only if the column of m

has an entry less than or equal to m in row j. Claim 6 then implies that $a_{m+2} = a_{n+2}$. Then $R_{m+2,j} = R_{m,j}$ except when $j = a_{m+2}$, and then $R_{m+2,j} = R_{m,j} + 1$. The same is true of $R_{n+2,j}$, so $A_{m+2} = A_{n+2}$, as claimed. □

Claim 8. There is a constant D such that $|R_{n,i+1} - R_{n,i}| \leq D$ for all positive integers n and i.

Proof of Claim 8. The problem statement tells us that, with finitely many exceptions, n is in column i if and only if $n + 1$ is in row i. Therefore, there is a constant L such that $|R_{n,i} - C_{n,i}| \leq L$ for all positive integers n and i. We will prove that $|C_{n,i+1} - C_{n,i}| \leq 2L$ for all positive integers n and i; since

$$R_{n,i+1} - R_{n,i} = (R_{n,i+1} - C_{n,i+1}) - (R_{n,i} - C_{n,i}) + (C_{n,i+1} - C_{n,i}),$$

this establishes Claim 8.

Since each row is sorted in increasing order, every entry in column $i + 1$ has an entry to its left; that is, $C_{n,i+1} \leq C_{n,i}$. Suppose, indirectly, that $C_{n,i} > C_{n,i+1} + 2L$ for $i < K$, and let $m - 1$ be the largest element in column i that is less than or equal to n. Then it is also true that

$$C_{m,1} > \cdots > C_{m,i} > C_{m,i+1} + 2L > \cdots > C_{m,K} + 2L.$$

Since $|R_{n,j} - C_{n,j}| \leq L$ for all n and j, then $R_{m,1}, \ldots, R_{m,i}$ are each greater than $R_{m,i+1}, \ldots, R_{m,K}$, and m is in the i^{th} row. By Claim 6, $m + 2, m + 4, \ldots$ are all in the first i rows, so only these rows are infinite, contradicting $i < K$. □

We are now ready to conclude the proof of this problem, as follows. Claim 8 implies that A_n can only take finitely many values as n ranges over rowlike numbers, of which there are infinitely many. Therefore, there must be rowlike numbers m, n such that $A_m = A_n$. But then Claim 7 implies $A_{m+2i} = A_{n+2i}$ for all positive integers i, so $a_{m+2i} = a_{n+2i}$. This means that at least one of the sequences a_1, a_3, a_5, … and a_2, a_4, a_6, … is eventually periodic, as claimed.

Problem 4 (393, 7, 8, 4, 10, 19, 39, 129; **4.854**); *proposed by Poland.* Let ABC be a triangle with $AB < AC < BC$. Let the incenter and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively.
Prove that $\angle KIL + \angle YPX = 180°$.

Solution. We start by introducing some additional notation. We let ℓ_X denote the line through X that is parallel to AC, and R and Z denote the intersections of ℓ_X with AB and AI, respectively. We then let ℓ_Z denote the line through Z that is parallel to AB, with S denoting the intersection of ℓ_Z with AC.

Since ℓ_X and AC are parallel, we have $\angle RZA = \angle ZAS$, and since AZ is the angle bisector of $\angle RAS$, we have $\angle RAZ = \angle ZAS$. Therefore, $ARZS$ is a rhombus.

Note that the distance between parallel lines AS and RZ is the diameter of the incircle, so the same distance holds for lines AR and SZ, showing that SZ must be tangent to the incircle and parallel to AB, so it must go through Y. We have that the incircle is tangent to the four sides of the rhombus, so I must be in the intersection of the diagonals, and hence $AI = IZ$.

Since L, I, K are midpoints of AB, AZ, AC, respectively, triangles LIK and BZC are similar, and thus $\angle LIK = \angle BZC$. Furthermore, $\angle CBP = \angle BAP = \angle XZA$, so

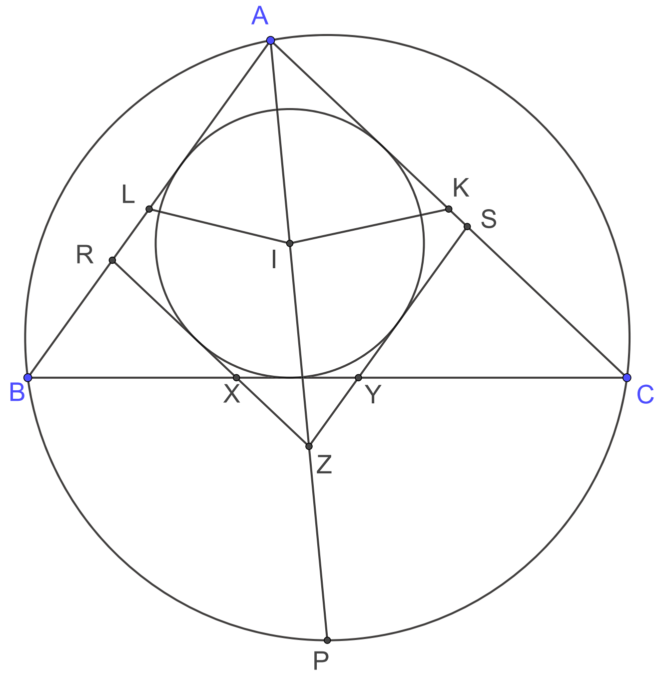

Figure 2 Illustration for Problem 4.

$BXZP$ is a cyclic quadrilateral, and thus $\angle XZB = \angle XPB$; similarly, we find that $CYZP$ is a cyclic quadrilateral, yielding $\angle CZY = \angle CPY$. Now

$$\angle CZB = \angle CZY + \angle YZX + \angle XZB,$$

which we can then rewrite as

$$\angle KIL = \angle CPY + \angle YZX + \angle XPB.$$

Therefore, we get

$$\angle KIL + \angle YPX = \angle CPY + \angle YPX + \angle XPB + \angle YZX = \angle CPB + \angle YZX.$$

Here $\angle YZX = \angle BAC$, and since $\angle BAC$ and $\angle CPB$ are supplementary angles, we get $\angle KIL + \angle YPX = 180°$, as desired.

Problem 5 (154, 11, 18, 1, 9, 3, 97, 316; **2.246**); *proposed by Hong Kong.* Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there is exactly one monster in each row except the first row and the last row, and that each column contains at most one monster.

Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common side. (He is allowed to return to a previously visited cell.) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over.

Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n^{th} attempt or earlier, regardless of the locations of the monsters.

Solution. The answer is $n = 3$.

First we explain why two attempts may not be enough. Initially, every cell in the second row is a potential monster location, so no matter his strategy, Turbo could encounter a monster on the second row on the first attempt. Assuming that he does, Turbo now knows the location of the monster in the second row, but every cell in the third row is a potential monster location, except the cell in the same column as the second row monster (see Figure 3). Therefore, no matter his strategy, the first cell he reaches in the third row could house a monster. Hence, Turbo cannot guarantee reaching the last row in just two attempts.

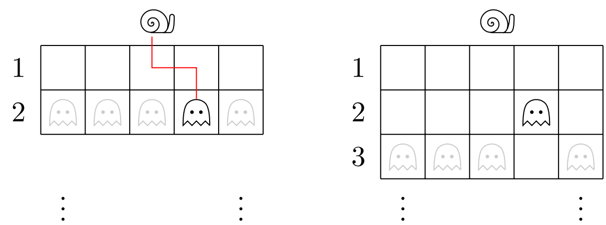

Figure 3 Turbo's first and second attempts may fail.

We now present a strategy that guarantees that Turbo will reach the last row in at most three attempts.

On his first attempt, he visits all cells in the second row in order to find the monster in that row. This ends his first attempt.

If the second row monster is not on the edge of the board, then it is in column k for some $2 \leq k \leq 2022$. In this case Turbo may follow either column $k-1$ or $k+1$ to the third row, cut back to column k, and continue to the last row on column k (see Figure 4). Since no row or column contains two monsters, the only place he could encounter a monster following this strategy is in the third row. However, there is only one monster in the third row, so one of these two paths must be free of monsters. Turbo is guaranteed to reach the last row on his second or third attempt.

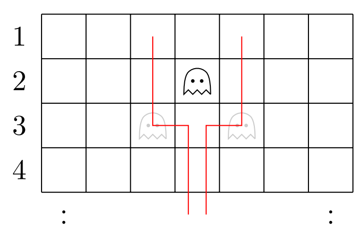

Figure 4 Turbo's strategy if the monster in the second row is not on the edge of the board.

Suppose now that the second row monster is on the edge of the board; without loss of generality, it is in the first column. In his second attempt, Turbo starts from the rightmost column and scans each row from right to left. When he scans the k^{th} row, he enters at the far right, scans left to the k^{th} column, then returns to the right and moves down to the $(k+1)^{\text{th}}$ row (as shown in Figure 5). If he does not encounter any monsters, this process will end in the 2023rd row, where he visits only one cell, and then he can safely move to the last row.

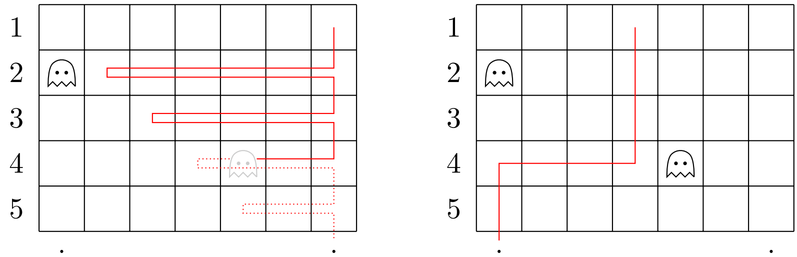

Figure 5 Turbo's strategy if the monster in the second row is on the edge of the board.

If he does encounter a monster, suppose it is in row r and column c, where $3 \leq r \leq c \leq 2023$. Since this was the first monster he encountered on his second attempt, there must not have been any monsters in column $c-1$, rows 2 through $r-1$. Thus, in his third attempt, Turbo may follow column $c-1$ down to row r without encountering a monster. As there cannot be two monsters in one row or one column, he may then follow row r to column 1, and finally follow column 1 to the last row, guaranteeing that he reaches the last row on the third attempt if not before.

Problem 6 (5, 1, 7, 2, 9, 27, 76, 482; **0.396**); *proposed by Japan.* Let $\mathbb{Q}$ be the set of rational numbers. A function $f : \mathbb{Q} \to \mathbb{Q}$ is called *aquaesulian* if the following property holds: for every $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \quad \text{or} \quad f(f(x) + y) = x + f(y). \tag{1}$$

Show that there exists an integer c such that for any aquaesulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r, and find the smallest possible value of c.

Solution. The answer is $c = 2$.

We start our proof by observing that if f is an aquaesulian function, then $x + f(x)$ is a fixed point of f for any rational number x. Indeed, letting $y = x$ in (1) we have

$$f(x + f(x)) = x + f(x). \tag{2}$$

Furthermore, we show that every aquaesulian function is injective. Suppose that $f(x) = f(y)$ for some rational numbers x and y. Then, from (1) and using (2), we get

- $x + f(x) = f(x + f(x)) = f(x + f(y)) = f(x) + y$, or
- $y + f(y) = f(y + f(y)) = f(f(x) + y) = x + f(y)$.

In both cases, we get $x = y$. Therefore, f is injective, as claimed.

We now prove that any aquaesulian function has at most two distinct rationals of the form $f(r) + f(-r)$; in fact, we prove that at most one such nonzero rational number may exist. Suppose that $f(r) + f(-r) \neq 0$ and $f(s) + f(-s) \neq 0$ for some rational numbers r and s (if no such r and s exist, then $f(x) + f(-x) = 0$ for all x). Our goal is to show that $f(r) + f(-r) = f(s) + f(-s)$.

Setting $x = r$ and $y = s$ in (1), we get $f(r + f(s)) = f(r) + s$ or $f(s + f(r)) = r + f(s)$. We can assume without loss of generality that

$$f(r + f(s)) = f(r) + s. \tag{3}$$

Let us now consider when $x = r + f(r)$ and $y = -r$ in (1). The first possibility, using also (2), yields

$$f(r + f(r) + f(-r)) = f(r + f(r)) + (-r) = r + f(r) + (-r) = f(r),$$

which we cannot have since f is injective and $f(r) + f(-r) \neq 0$. Therefore, the second possibility in (1) must hold, and thus

$$f(f(r + f(r)) + (-r)) = r + f(r) + f(-r).$$

Here the left side equals $f(f(r))$ by (2), thus we get

$$f(f(r)) = r + f(r) + f(-r). \tag{4}$$

Now take $x = r + f(s)$ and $y = -s$ in (1). We can again rule out the first possibility, since (3) would imply

$$f(r + f(s) + f(-s)) = f(r + f(s)) + (-s) = f(r),$$

contradicting the fact that f is injective as $f(s) + f(-s) \neq 0$. Therefore, we must have

$$f(f(r + f(s)) + (-s)) = r + f(s) + f(-s),$$

which using (3) becomes

$$f(f(r)) = r + f(s) + f(-s).$$

Comparing this to (4), we arrive at $f(s) + f(-s) = f(r) + f(-r)$, as desired, completing the proof that $f(x) + f(-x)$ can take at most two different values when $x \in \mathbb{Q}$.

To see that $c = 2$, we now exhibit an aquaesulian function f that has exactly two different rationals of the form $f(r) + f(-r)$.

Let $f(x) = \lfloor x \rfloor - \{x\}$, where $\lfloor x \rfloor$ is the integer part of x and $\{x\}$ is the fractional part of x. It is an easy exercise to verify that f is an aquaesulian function; namely, if $\{x\} \geq \{y\}$, then $f(x + f(y)) = f(x) + y$, and if $\{y\} \geq \{x\}$, then $f(x + f(y)) = f(x) + y$. For example, the second assertion follows from the fact that

$$f(x) + y = \lfloor x \rfloor - \{x\} + \lfloor y \rfloor + \{y\},$$

so when $\{y\} \geq \{x\}$, we have $\lfloor f(x) + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ and $\{f(x) + y\} = \{y\} - \{x\}$, and therefore

$$f(f(x) + y) = \lfloor x \rfloor + \lfloor y \rfloor - (\{y\} - \{x\}) = x + f(y).$$

For this function, we have

$$f(r) + f(-r) = \lfloor r \rfloor + \lfloor -r \rfloor - (\{r\} + \{-r\}).$$

Note that $\lfloor r \rfloor + \lfloor -r \rfloor$ equals 0 when r is an integer and -1 otherwise, and $\{r\} + \{-r\}$ equals 0 when r is an integer and 1 otherwise. Therefore, $f(r) + f(-r)$ is either 0 (when r is an integer) or -2 (when r is not an integer), so there are exactly two rational numbers that can be written in the form $f(r) + f(-r)$, as claimed.

Summary. We present the problems and solutions to the 65th Annual International Mathematical Olympiad.

BÉLA BAJNOK (MR Author ID: 314851) is a Professor of Mathematics at Gettysburg College and the Director of the American Mathematics Competitions program of the MAA.

JOHN BERMAN (MR Author ID: 1284960) is an independent mathematician, the Academic Director of the Math Olympiad Program, and the Team Leader of the United States IMO team.

ENRIQUE TREVIÑO (MR Author ID: 894315) is a Professor of Mathematics at Lake Forest College and the co-editor in chief of the USAMO.