Explicit Burgess inequalities for cubefree moduli

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Dirichlet Character

Let n be a positive integer.
$\chi : \mathbb{Z} \to \mathbb{C}$ is a Dirichlet character $\bmod n$ if the following three conditions are satisfied:

- $\chi(a+n) = \chi(a)$ for all $a \in \mathbb{Z}$.
- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$.
- $\chi(a) \neq 0$ if and only if $\gcd(a, n) = 1$.

The Legendre symbol is an example of a Dirichlet character.

Pólya–Vinogradov

Let χ be a Dirichlet character to the modulus $q > 1$. Let

$$S(\chi) = \max_{M,N} \left| \sum_{n=M+1}^{M+N} \chi(n) \right|$$

The Pólya–Vinogradov inequality (1918) states that there exists an absolute universal constant c such that for any Dirichlet character $S(\chi) \leq c\sqrt{q} \log q$.

Under GRH, Montgomery and Vaughan showed that $S(\chi) \ll \sqrt{q} \log \log q$.

Paley showed in 1932 that there are infinitely many quadratic characters such that $S(\chi) \gg \sqrt{q} \log \log q$.

Burgess Inequality

Theorem (Burgess, 1962)

Let χ be a primitive character $\bmod q$, where $q > 1$, r is a positive integer and $\epsilon > 0$ is a real number. Then

$$|S_\chi(M,N)| = \left| \sum_{M<n\leq M+N} \chi(n) \right| \ll N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon}$$

for $r = 1, 2, 3$ and for any $r \geq 1$ if q is cubefree, the implied constant depending only on ϵ and r.

Quadratic Case for Burgess

Theorem (Booker, 2006)

Let $p > 10^{20}$ *be a prime number* $\equiv 1 \pmod 4$, $r \in \{2, \ldots, 15\}$ *and* $0 < M, N \leq 2\sqrt{p}$. *Let* χ *be a quadratic character* $\pmod p$. *Then*

$$\left| \sum_{M \leq n < M+N} \chi(n) \right| \leq \alpha(r) p^{\frac{r+1}{4r^2}} (\log p + \beta(r))^{\frac{1}{2r}} N^{1-\frac{1}{r}}$$

where $\alpha(r), \beta(r)$ *are given by*

r	$\alpha(r)$	$\beta(r)$	r	$\alpha(r)$	$\beta(r)$
2	1.8221	8.9077	9	1.4548	0.0085
3	1.8000	5.3948	10	1.4231	-0.4106
4	1.7263	3.6658	11	1.3958	-0.7848
5	1.6526	2.5405	12	1.3721	-1.1232
6	1.5892	1.7059	13	1.3512	-1.4323
7	1.5363	1.0405	14	1.3328	-1.7169
8	1.4921	0.4856	15	1.3164	-1.9808

Explicit Burgess

Theorem (Iwaniec-Kowalski-Friedlander)

Let χ be a non-principal Dirichlet character mod p (a prime). Let M and N be non-negative integers with $N \geq 1$ and let $r \geq 2$, then

$$|S_\chi(M,N)| \leq 30 \cdot N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

Theorem (ET, 2015)

Let p be a prime. Let χ be a non-principal Dirichlet character mod *p. Let M and N be non-negative integers with $N \geq 1$ and let r be a positive integer. Then for $p \geq 10^7$, we have*

$$|S_\chi(M,N)| \leq 2.74 N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

Theorem (Francis, 2021)

Improvements for $2 \leq r \leq 10$ with $N < 2p^{5/8}$.

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Francis Table

r	$p_0 = 10^5$	$p_0 = 10^6$	$p_0 = 10^7$	$p_0 = 10^8$	$p_0 = 10^9$	$p_0 = 10^{10}$
2	3.7125	3.4682	3.3067	3.1980	3.1259	3.0679
3	2.7979	2.6371	2.5131	2.4318	2.3776	2.3358
4	2.4157	2.2980	2.2022	2.1513	2.0994	2.0613
5	2.1801	2.0981	2.0273	1.9755	1.9419	1.9084
6	2.0874	2.0037	1.9424	1.8962	1.8353	1.8054
7	1.8948	1.8454	1.8087	1.7820	1.7561	1.7291
8	1.7993	1.7609	1.7306	1.7093	1.6894	1.6696
9	1.7266	1.6963	1.6692	1.6492	1.6323	1.6186
10	1.6720	1.6411	1.6175	1.5991	1.5845	1.5727

r	$p_0 = 10^{11}$	$p_0 = 10^{12}$	$p_0 = 10^{13}$	$p_0 = 10^{14}$	$p_0 = 10^{15}$	$p_0 = 10^{16}$
2	3.0280	2.9997	2.9790	2.9635	2.9515	2.9421
3	2.3025	2.2782	2.2600	2.2461	2.2351	2.2263
4	2.0329	2.0117	1.9956	1.9831	1.9733	1.9654
5	1.8831	1.8638	1.8487	1.8367	1.8272	1.8194
6	1.7825	1.7646	1.7503	1.7388	1.7294	1.7216
7	1.7081	1.6914	1.6779	1.6669	1.6577	1.6500
8	1.6501	1.6345	1.6219	1.6112	1.6023	1.5946
9	1.6029	1.5882	1.5762	1.5661	1.5575	1.5501
10	1.5629	1.5499	1.5384	1.5287	1.5205	1.5134

r	$p_0 = 10^{17}$	$p_0 = 10^{18}$	$p_0 = 10^{19}$	$p_0 = 10^{20}$	$p_0 = 10^{50}$	$p_0 = 10^{75}$
2	2.9345	2.9282	2.9230	2.9185	2.8752	2.8658
3	2.2190	2.2128	2.2076	2.2029	2.1503	2.1368
4	1.9590	1.9537	1.9493	1.9455	1.9094	1.9011
5	1.8130	1.8077	1.8033	1.7996	1.7689	1.7630
6	1.7151	1.7097	1.7051	1.7012	1.6715	1.6668
7	1.6435	1.6380	1.6333	1.6292	1.5986	1.5947
8	1.5883	1.5828	1.5779	1.5738	1.5986	1.5382
9	1.5439	1.5384	1.5336	1.5294	1.4959	1.4925
10	1.5072	1.5019	1.4972	1.4930	1.4581	1.4548

Some Applications of the Explicit Estimates

- Booker (2006) computed the class number of a 32-digit discriminant using an explicit estimate of a character sum.
- McGown proved (2012) that there is no norm-Euclidean cubic field with discriminant $> 10^{140}$. Later (2017) with Lezowski they proved it for discriminants over 10^{100}.
- Bagger, Booker, Kerr, McGown, Starichkova, and Trudgian were able to classify all norm-Euclidean cubic fields. (2025)
- Explicit bound on the least prime primitive root done by Cohen, Oliveira e Silva and Trudgian (2016).
- Elkies used explicit Burgess inequalities to prove a particular elliptic curve had rank 16 (2025).
- Johnston, Ramaré, Trudgian used Burgess inequalities to improve explicit bounds $L(1, \chi)$.

Burgess Inequality

Theorem (Burgess, 1962)

Let χ be a primitive character $\bmod q$, where $q > 1$, r is a positive integer and $\epsilon > 0$ is a real number. Then

$$|S_\chi(M,N)| = \left| \sum_{M<n\le M+N} \chi(n) \right| \ll N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon}$$

for $r = 1, 2, 3$ and for any $r \geq 1$ if q is cubefree, the implied constant depending only on ϵ and r.

Composite Burgess

Theorem (Jain-Sharma, Khale, Liu, 2021)

Let χ be a primitive character mod q with $q \geq e^{e^{9.594}}$. Then, for $N \leq q^{\frac{5}{8}}$,

$$|S_\chi(M,N)| \leq 9.07\sqrt{N}q^{\frac{3}{16}}\log^{\frac{1}{4}}(q)\left(2^{\omega(q)}\tau(q)\right)^{\frac{3}{4}}\left(\frac{q}{\phi(q)}\right)^{\frac{1}{2}},$$

where $\tau(q)$ is the number of divisors of q and $\omega(q)$ is the number of distinct prime factors of q.

This is the $r = 2$ case of the Burgess inequality made explicit for all large enough moduli q.

Burgess for cubefree moduli and $r \geq 3$

Theorem (H-L-L-M-T)

Let $r \geq 2$ be an integer and χ be a primitive Dirichlet character modulo q. Let $m_r(q) = \min\left\{\tau_{2r}(q), \left(\frac{\tau(q)}{2}\right)^{2r-1}, \frac{q}{2r}\right\}$. Let $C(r)$ be defined as in Table 1. Let
$a(r) = 2\log 2\,(3.0758r + 1.38402\log(4r) - 1.5379)$. Then, for $q \geq \max\{10^{1145}, e^{e^{a(r)}}\}$, if $r = 2$ or q is cubefree, we have

$$|S_\chi(M,N)| \leq C(r) N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}-\frac{1}{2r^2}} \left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

Furthermore, we have a constant $D(r)$ from Table 1 such that

$$|S_\chi| \leq (D(r)+o(1)) N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}-\frac{1}{2r^2}} \left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

Another Burgess inequality for cubefree with less restrictions on the size of q

Theorem (H-L-L-M-T-2)

Let χ be a primitive Dirichlet character modulo q . Let $C(r)$ be defined as in Table 1. Then, for $q \geq \max\{10^{1145}, 2^{4r-2}\}$, if $r = 2$ or q is cubefree, we have

$$|S_\chi(M,N)| \leq C(r) N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}} \left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

Table of constants

r	$C(r)$	$D(r)$
2	15.219	8.362
3	5.359	4.581
4	3.671	3.396
5	2.953	2.811
6	2.549	2.462
7	2.290	2.229
8	2.108	2.063
9	1.973	1.938
≥ 10	1.869	1.841

Table: Constants in the Burgess inequality for values of r.

Key Inequality to prove Burgess Inequalities

Theorem (Weil-type inequality for prime p)

Let $r \geq 2$ be an integer and $B \geq 2$ be real. Let χ be a primitive Dirichlet character $\bmod p$*, then*

$$\sum_{x=1}^{p} \left| \sum_{1 \leq b \leq B} \chi(x+b) \right|^{2r} < (2r-1)!!p\,B^r + (2r-1)\sqrt{p}\,B^{2r}.$$

Theorem (Weil-type inequality)

Let $r \geq 2$ and q be positive integers such that $r = 2$ or q is cubefree. Let χ be a primitive Dirichlet modulo q. Let $B \geq 2$ be a real number. Then

$$\sum_{x=1}^{q} \left| \sum_{1 \leq b \leq B} \chi(x+b) \right|^{2r} \leq \frac{r^{2r}}{r!} B^r q + 2r(4r)^{\omega(q)} B^{2r} m_r(q) \sqrt{q}.$$

Celebrating

Thank you!