Walking on Numbers and a Self-Referential Formula

Enrique Treviño

Lake Forest College

Camp Conway, Harvey Mudd College
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Coauthors for Walking on Numbers

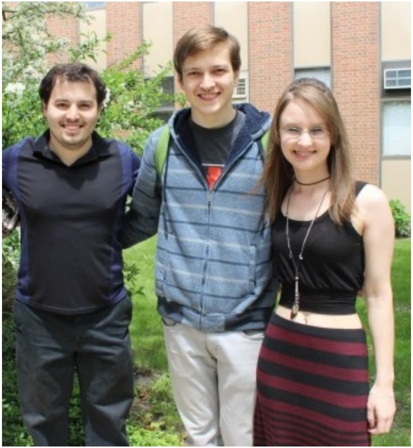


Figure: Kevin Kupiec, Marina Rawlings and me.

Background

- "Walking on Real Numbers" by Aragón, Bailey, Borwein and Borwein.
- Consider a number in base 4. For example π. In base 4,

$$\pi = 3.0210033312222020201122030020310301\ldots,$$

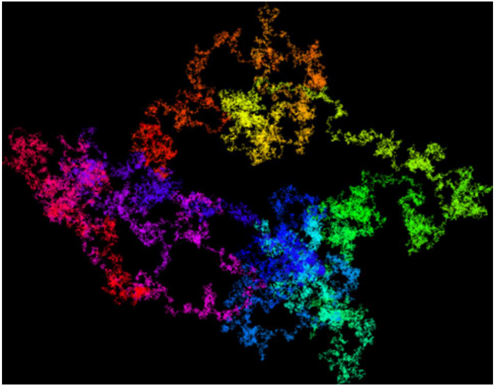
because

$$\pi = 3 + 0\left(\frac{1}{4}\right) + 2\left(\frac{1}{4^2}\right) + 1\left(\frac{1}{4^3}\right) + \ldots$$

- We start at the origin in the Cartesian plane. We move a unit to the right whenever we hit a digit 0, we move a unit up whenever we hit a digit 1, we move a unit left whenever we hit a digit 2 and we move a unit down if we hit a digit 3.

Walking on π

Walking on the first 100 billion digits of π reveals the following picture:



Easier Example

Walking on the number 419636198 which can be rewritten in base 4 as

$$121000302033212_4.$$

Inspiration

10490122716774994374866192805654486016 17567358491560876166848380843144358447
25287555162924702775955557045371567931 30587832477297720217708181879659063736
57674879814228013285920278610192581409 57135748704712290267465151312805954195
3997504202061380373822338959713391954

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16122269626942909129404900662735492142 29880755725468512353395718465191353017
34881431401750453996944547935301206438 33272670970079330526292030350920973600
45095545613659664932507839146477284016 23856513742952945308961226815274887561
5658076162410788075184599421938774835

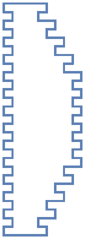
Project

- For each letter of the alphabet, find a rational that satisfies that if you random walk through that rational, you get the letter of the alphabet.
- Build a computer program that can use the information above to figure out the rational that works for a particular phrase.

Alphabet

How do you find the rational for a particular letter?

- Find a string of digits that “spell out” your letter in such a way that you end up where you started. For example, for the letter “D”, it would be 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 3, 0, 1, 2, 0, 3, 0, 3, 2, 0, 0, 3, 2, 0, 0, 3, 2, 0, 3, 2, 0, 0, 3, 2, 0, 3, 2, 3, 0, 2, 3, 0, 3, 2, 0, 0, 3, 2, 2, 3, 0, 0, 3, 2, 2, 1, 3, 0, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 2, 2, 3, 0, 0, 1, 3, 2, 2, 3, 0, 0, 3, 2, 2, 0, 3, 2, 3, 0, 2, 3, 0, 2, 3, 2, 0, 3, 2, 2, 0, 3, 2, 0, 3, 2, 2, 0, 3, 2, 2, 0, 3, 2, 1, 2, 0, 3, 2.



- A string of digits that spells "D", is 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 3, 0, 1, 2, 0, 3, 0, 3, 2, 0, 0, 3, 2, 0, 0, 3, 2, 0, 3, 2, 0, 0, 3, 2, 0, 3, 2, 3, 0, 2, 3, 0, 3, 2, 0, 0, 3, 2, 2, 3, 0, 0, 3, 2, 2, 1, 3, 0, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 2, 2, 3, 0, 0, 1, 3, 2, 2, 3, 0, 0, 3, 2, 2, 0, 3, 2, 3, 0, 2, 3, 0, 2, 3, 2, 0, 3, 2, 2, 0, 3, 2, 0, 3, 2, 2, 0, 3, 2, 2, 0, 3, 2, 1, 2, 0, 3, 2.
- Consider the dot product with $\{1/4, 1/4^2, 1/4^3, \ldots, \ldots\}$. In our example we'd have

$$1\left(\frac{1}{4}\right) + 0\left(\frac{1}{4^2}\right) + 1\left(\frac{1}{4^3}\right) + \ldots =$$

$$\frac{638477938204395103621734866125368051500588535748453547158965451495641479466272100636854259724898698532312741670451981081526131897015418 3}{232588391774594204975783618524161450993165235419941779290076863737804572196287335464381136228404340979444006914005176938731072521156689 92}$$

One issue with just finding the rational as above, is that the random walk will now continue indefinitely to the right as the expansion ends with infinitely many zeroes. To fix this, we consider the number of digits in the representation of the letter and then do a geometric series expansion.
For example with the letter "D", it has 227 digits. Let the rational representation be x. Then the rational that loops itself over and over would be

$$x + 4^{-227}x + \left(4^{-227}\right)^2 x + \ldots = \frac{1}{1 - 4^{-227}}x.$$

Example: Numerator has 1622 digits

5098346085560368565970377351327456875608351509202691891184744019248361413189995625910292004871423491514496586090821801996357571814233573004803933565864457756266
5514549456146030867146265016638795180352883908617046400108387068689833516264465154656545355744980649903343455127979280483395861705152024270453574051045396532325
8866578654621347804302792144838805971890078124921963288245010854678336608105125529919383195593968095210946909336152836216808809350699515767351280144423998059097
5403215219385514427165923679252889597262433147758383713031121013070251600684593504917280538327730533027156982226750371711412047394031336322134944752522173141088
0301105113105617485792045858425812983001463302251136215133121064906525008286913109089843675180462007721156654013806562630303524530830590063464306170361388931707
7454642152822846981660097738879070338916519208448085824119272323890560574501240084931047284689547119384603128406532492042244568123862825520191810754771229819091
9770032737965574413810794166279855218313304039256469687216427393503540960937102907460982908923976071288811413779889487179927262744363276099860841111789320236490
5200276322577489979946225797152440429079006870579726503338324399682853279452501801826674664268705703957013934680807504303610041619910243593113978925045237415740
6830801226163365021924475784340830913140812650579517454521881340871641928458147286715129938330821529368974366858579020899436886987349314128824767826741532048977
9259828426853562147894886781517672340196428458447250095462436356104255344150268091211327184700564738481898777824081479240692888059374464895730885135406157113087
2538024758545530661546

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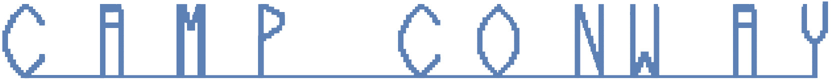
8904645560409693555435700788795217436298618982709442764184532347058055398732911242425396639644929296048173576161343351494686413739928750956093815208641594302722
7125600465992625398105401067452639500736888223420869447496895425935984485227985056016965269598116938095013015407958468655364170856188254886796695919382245281966
9536428576367315482295084971156446941515309330004340471587425623688260444158587276473116933881186087483692268306161800783582674894130376094088880083299001874567
3645522833461514917827849769657064000564917991905645932098156676671738102031535409634597877364838789255792995369693900154263668940661449261148706329024961506748
1938581717940756075744583841000666832609509558820969101616165413170630407561558227512247646840376124319640182150563712680265598764539821795568756325608183669959
9377038842257712732393758779557645322439446848984166741703144934045875434614490013526353278096993657265424693124824178284262330171003289361495251034947459870818
1498753614008340723341971618139257112389567692361067166402111705897198009705886576599268870612539800194400382966615845417702052835329675248512871741656813440095
6575001590048032689897213485389334649752652693449550000547383317197923426159617483783614597987625636615882263466263942257048765415388753143926639103116711826891
7307975265257563040861769416359316823707338749490027532252600341578268693908665315035714817724243783659762027410754061871608515202516467892351412518033698067566
0246011625690933152708210208179210111626951765229045838055877160494121704146037915154437716377971137690343807830770454017840530890861616066048264892259786394364
9771889186993713709055.

Picture



Suppose that for every letter α (a variable representing an uppercase letter from the English alphabet), you find a rational r_α and an integer n_α, such that the walk of r_α with n_α steps spells the letter α in such a way that the last step ends at the origin. We will also define $r_{blank} = n_{blank} = 0$, i.e., representing a blank space. Finally, suppose the base of each letter is at most w, i.e., the length of a blank space is w.

Theorem

Suppose we are given a sentence σ which we'll write as $\sigma = \alpha_1\alpha_2\alpha_3\cdots\alpha_k$, where α_i is a letter or a space. Let

$$n = \sum_{i=1}^{k} n_{\alpha_i} + 2(k-1)w, \tag{1}$$

and

$$r = \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_j}+w)\right)}} + \frac{2\cdot 4^{w}\left(1-\frac{1}{4^{(k-1)w}}\right)}{3\cdot 4^{\left(\sum_{j=1}^{k}(n_{\alpha_j}+w)\right)}}. \tag{2}$$

Then a walk on r of length n spells out σ. Furthermore, a walk on $\frac{4^n}{4^n-1}r$ of any length $m \geq n$ spells out σ.

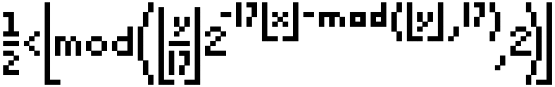
Coauthor on a Self-Referential Formula

Figure: Margaret Fortman and me.

Tupper's self-referential formula

$$\frac{1}{2} < \left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \mathrm{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor$$

Graph of the above equation for $0 \le x \le 105$ and $k \le y \le k + 17$ is



for k equal to
4858450636189713423582095962494202044581400587983244549483093085061934704708809
92845064476986552436484999724702491511911041160573917740785691975432657185544205721
04457358836818298237541396343382251994521916512843483329051311931999535024137587652
39264874613394906870130562295813219481113685339535565290850023875092856892694555974
281546386510730049106723058933586052544096664351265349363643957125556569593681518433
48576052669401612512669514215505395545191537854575257565907405401579290017659679654
80064427829131488548259914721248506352686630476300

Project

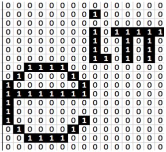
For a given sentence, find the integer k such that the graph of Tupper's formula looks like that sentence for $0 \leq x \leq 105$ and $k \leq y \leq k+17$.

Example

Plot of Tupper's formula for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$ when k is

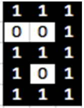
24423375057634273873575858117465207144938121226248217479023924030161082657913433430825706268459998045719387304611581033828548214011854466382174297483978631035132877153850565726004763476287273161306962862050524181719834772905989380587629345613928425991474426124418944086589636253079524941548171672160527454059369910848156471504464523579591740567200567168101347784461629002716147189238472062171330230568434148182551792072073757257894787163228562225290858628 48

How to get it done



0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	1	1	1	1	1
0	0	0	0	0	0	0	0	1	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	1	0	1	0	1	0
0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

As the previous project, the key is figuring out how to do a letter first.



1	1	1
0	0	1
1	1	1
1	0	1
1	1	1

Letter a

1	1	1
0	0	1
1	1	1
1	0	1
1	1	1

The binary number for the letter a is 11111 10101 11101
Multiply by 2^{17} to move column to the right.
Formula for the lowercased a is:

$$17((1+2+4+16)+(1+4+16)2^{17}+(1+2+4+8+16)2^{34})$$

Theorem

Let $k = 17k'$ for a nonnegative integer $k' < 2^{106\times 17}$. Suppose we write k' in binary as follows:

$$k' = \sum_{m=0}^{105}\sum_{n=0}^{16} a_{17m+n}2^{17m+n}.$$

Then

$$\left\lfloor \operatorname{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x\rfloor - \operatorname{mod}(\lfloor y\rfloor,17)}, 2\right)\right\rfloor = a_b,$$

for $b = 17\lfloor x\rfloor + \operatorname{mod}(\lfloor y\rfloor, 17)$.
Therefore, the point (x, y) is painted whenever $a_b = 1$ and not painted when $a_b = 0$, i.e., it depends only on the binary expansion of k'.

Theorem

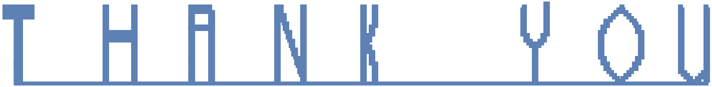
Given a sentence $\sigma = \alpha_1\alpha_2\cdots\alpha_k$, where α_i represents a single letter or a blank space and $k \leq 63$, we use the following formula to figure out the value of k for the range where the plot of Tupper's formula is σ:

$$\sum_{i=1}^{\min(21,k)} 2^{85(i-1)+12} f(\alpha_i) + \sum_{i=22}^{\min(42,k)} 2^{85(i-22)+6} f(\alpha_i) + \sum_{i=43}^{k} 2^{85(i-43)} f(\alpha_i). \tag{3}$$

Proof.

Each letter fits in a block of width 5 and height 5. To move from one letter to the next (to the right), we need to multiply by $2^{17\times 5} = 2^{85}$. This is where the 85's in the exponents come from. The reason we add 12 and 6 (depending on how many letters we have) is because the first row consists of numbers in the top "strip" ($k + 12 \leq y < y + 17$), so we have to multiply by 2^{12} to move upwards. The numbers in the middle strip ($k + 6 \leq y \leq y + 11$) need a shift of 2^6, and the bottom row needs no translation. The formula follows. $\square$

2604492038498829664517035139638426870137917317601914921681571572073092
8695649142399363187609280194294167878123410715564434829372484954369221
6688674071161566295182034925025124500569335817787439872314183988703371
6649770348100268419393881844091449254145085981313775888184477056



2448384190858685068867681259343425875183951488272013453075082431177592 0962050203904557003580654935217327434523590196726040466969501801090605
5664172807097345932129875992132606806463166306220789443630207743373850 7810762263639386835401550161949045105588754258631414471096220189442622
1665427118990256510546429000993755547847039756192941247474274244110787 8660478187344039509845541294428633493975708789645581674249072107946910
2765408044063942823929158894523762592434026978546624093591607103258197 4450638724117458467731275518440553284813684834902522605636902712794432
6983720681160278275882713626711046114853618093566074424105013908484306 7404052939064793858614545855382152074663980612417168491376157942144755
1424213635574427532612628956112547281010460985326871870253127060031406 9020366919487757012852917356230205755317365507728854738429219183094869
3695972821843598905429765758338285668885349223378577986330045089333457 9166101811912895915931607121962558396750697854916244499605957383922242
2087768224286810034216057699362187013656283147052173522331783042174874 4140094804118333840723288129692077185337544126726009078432030025686771
5239809488805445016460285761824177350795554860429732609262786222124676 6008948260384862093913280146413160526444220022

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3819479337739529167329838484906512027721012110801860682548053881724174 2920102672707614237693622767847895409616985207326646402685739366166208
4840626414213587363092837135985370821617952963431759840705122747193031 7627890537469797349469887395417031631058274040351838367832032124521106
7194847614828901800156489387408054011676810457486834475529542970498905 7255432583059392620413461586291555620523049770255248031290472335172139
2412944217588056092569134224444251794977065504945068111268913987049510 5866348693513857856730829398652779264556390283916331450215630875473236
7905955243200201225507949865086092749749275242859745654420665019701251 9591247855104320397246547508188151581997031018121793852983572059720856
2703336970436575992101116604676953965084910204328651037047629049824332 4556534083393752340059349194724035480526710224113881565618356297919255
1847086748814440011395684253058468278210013519149265023328261049164967 8735218721891047967321260999237484686693722176492735043666916683333478
2535713206663022363291361908461665812355382922360345773026535557687152 4820925610319529593771464824025674303842751783960198988339742665429620
7718565411907925539753987771468284452619059692514367146906203118121384 3353885892766513770146305721907347439246751 45