EXPLICIT BURGESS INEQUALITIES FOR CUBEFREE MODULI

ELCHIN HASANALIZADE, HUA LIN, GREG MARTIN, ANDRADIS LUNA MARTÍNEZ, AND ENRIQUE TREVIÑO

Abstract. Burgess proved that for χ_q a primitive Dirichlet character modulo q with q cubefree, $\left|\sum_{M<n\le M+N}\chi_q(n)\right| \ll N^{1-\frac{1}{r}}q^{\frac{r+1}{4r^2}+\epsilon}$ for all integers $r\ge 1$. More recently, explicit versions with prime moduli q were computed by Booker, McGown, Treviño, and Francis, with applications to finding the least k-th power residue, and bounding the size of Dirichlet L-functions just to name a few. Jain-Sharma, Khale, and Liu proved an explicit estimate for $r=2$. We improve their explicit constant for $r=2$ and compute an explicit Burgess bound for cubefree q for $r\ge 3$.

1. Introduction

Let q be a positive integer. Given a primitive Dirichlet character $\chi \pmod q$ and integers M and $N\ge 1$, we consider the character sum

$$S_\chi(M,N) = \sum_{M<n\le M+N}\chi(n).$$

Burgess, in a series of papers [2, 3, 4, 5, 6], proved

$$|S_\chi(M,N)| \ll N^{1-\frac{1}{r}}q^{\frac{r+1}{4r^2}+\varepsilon},$$

for $r=2,3$ and for any $r\ge 4$ when q is cubefree. In recent years, there has been progress in obtaining explicit estimates for $S_\chi(M,N)$. In the case when q is prime, Booker [1] proved a result for χ quadratic, and McGown [13] and Treviño [16] made improvements for general χ. The current best result for q prime and $2\le r\le 10$ is due to Francis [9]. An example of an explicit Burgess inequality for $q=p$ prime is the following uniform bound (from [16])

$$|S_\chi(M,N)| \le 2.74N^{1-\frac{1}{r}}p^{\frac{r+1}{4r^2}}(\log p)^{\frac{1}{r}},$$

for all integers $r\ge 2$ and for all primes $p\ge 10^7$.

These explicit results have been used to determine the class number of a quadratic field of large discriminant [1], to bound the least quadratic non-residue modulo a prime [9, 16], to classify Norm-Euclidean cyclic fields [13], to bound the least primitive root modulo a prime [7], to give an explicit bound for $L(1,\chi)$ in the case of quadratic characters [12], to compute the rank of an elliptic curve with large rank [8], among other applications.

In the case of composite moduli q, the only explicit result is the following due to Jain-Sharma, Khale, and Liu [11].

Theorem A. Let χ be a primitive character mod q with $q\ge e^{e^{9.594}}$. Then, for $N\le q^{\frac{5}{8}}$,

$$|S_\chi(M,N)| \le 9.07\sqrt{N}q^{\frac{3}{16}}\log^{\frac{1}{4}}(q)\left(2^{\omega(q)}\tau(q)\right)^{\frac{3}{4}}\left(\frac{q}{\phi(q)}\right)^{\frac{1}{2}},$$

where $\tau(q)$ is the number of divisors of q and $\omega(q)$ is the number of distinct prime factors of q.

In other words, the only explicit result for composite moduli q is for the case $r = 2$. We improve this result for $r = 2$ and q large enough, and we also generalize the result to all $r \geq 3$ when q is cubefree.

To state our theorem, we use the following standard labels of arithmetic functions: let $\omega(q)$ denote the number of distinct prime factors of q, let $\phi(q)$ be Euler's totient function, and let $\tau_k(q)$ denote the number of ways to write q as an ordered product of k integers, so that $\tau(n) = \tau_2(n)$ is the ordinary number-of-divisors function. Also define

$$m_r(q) = \min\left\{\tau_{2r}(q), \left(\frac{\tau(q)}{2}\right)^{2r-1}, \frac{q}{2r}\right\}. \tag{1}$$

Theorem 1.1. *Let $r \geq 2$ be an integer and χ be a primitive Dirichlet character modulo q. Let $C(r)$ be defined as in Table 1. Let $a(r) = 2\log 2\,(3.0758r + 1.38402\log(4r) - 1.5379)$. Then, for $q \geq \max\{10^{1145}, e^{e^{a(r)}}\}$, if $r = 2$ or q is cubefree, we have*

$$|S_\chi(M,N)| \leq C(r) N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}-\frac{1}{2r^2}} \left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

Furthermore, as $q \to \infty$, we have a constant $D(r)$ from Table 1 such that

$$|S_\chi(M,N)| \leq (D(r)+o(1)) N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}-\frac{1}{2r^2}} \left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

r	$C(r)$	$D(r)$
2	15.219	8.362
3	5.359	4.581
4	3.671	3.396
5	2.953	2.811
6	2.549	2.462
7	2.290	2.229
8	2.108	2.063
9	1.973	1.938
$\geq$10	1.869	1.841

TABLE 1. Constants in the Burgess inequality for values of r.

The theorem above requires q to be very large, so we have the following theorem which is weaker asymptotically but works for smaller values of q.

Theorem 1.2. *Let χ be a primitive Dirichlet character modulo q . Let $C(r)$ be defined as in Table 1. Then, for $q \geq \max\{10^{1145}, 2^{4r-2}\}$, if $r = 2$ or q is cubefree, we have*

$$|S_\chi(M,N)| \leq C(r) N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}} \left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

To prove Theorems 1.1 and 1.2 we need the following explicit Weil-type inequality, which we consider of independent interest.

Theorem 1.3 (Weil-type inequality)**.** *Let $r \geq 2$ and q be positive integers such that $r = 2$ or q is cubefree. Let χ be a primitive Dirichlet modulo q. Let $B \geq 2$ be a real number. Then*

$$\sum_{x=1}^{q} \left| \sum_{1 \leq b \leq B} \chi(x+b) \right|^{2r} \leq 2r(4r)^{\omega(q)} B^{2r} m_r(q) \sqrt{q} + \frac{r^{2r}}{r!} B^r q.$$

The paper is organized as follows. We prove Theorem 1.3 in Section 2. In Section 3 we describe the overall plan for the proofs of our main theorems following the approach for the explicit Burgess inequality detailed in [10, Theorem 12.6]. We prove other technical lemmas in Section 4, some important bounds in Section 5, and we prove our main theorems in Section 6.

Notation. Most of the notation in the paper is standard, for example, $\phi(n)$ is Euler's totient function, for a finite set S, $\#S$ is the number of elements of S, $\lfloor x \rfloor$ is the floor of x, for a pair of integers a, b, $(a, b) = \gcd(a, b)$. We also use Landau's O notation, i.e., $f(x) = O(g(x))$ if there exists a positive constant C such that $|f(x)| \leq C|g(x)|$ for large enough x. A notation we use that is less well-known is O^*. We say that $f(x) = O^*(g(x))$ if for all $x \geq x_0$, $|f(x)| \leq |g(x)|$.

2. Explicit Weil-type inequality

This section can be read independently from the rest of the paper. Let $r \geq 2$ be an integer, $B \geq 2$ be a real number, and

$$m_r(q) = \min\left\{ \tau_{2r}(q), \left(\frac{\tau(q)}{2}\right)^{2r-1}, \frac{q}{2r} \right\}. \tag{2}$$

Furthermore, given integers $b_1, b_2, \ldots, b_{2r}$, we define

$$A_j = \prod_{\substack{1 \leq i \leq 2r \\ i \neq j}} (b_i - b_j).$$

Burgess [2, Lemma 7], [5, Lemma 8] proved the following analogue of Weil's inequality that holds for composite moduli:

Lemma 2.1 (Burgess)**.** *Let $b_1, b_2, \ldots, b_{2r}$ be integers such that at least $r+1$ of them are distinct. Let $f_1(x) = (x-b_1)(x-b_2)\cdots(x-b_r)$ and $f_2(x) = (x-b_{r+1})(x-b_{r+2})\cdots(x-b_{2r})$. Let χ be a primitive Dirichlet mod q. If $r = 2$ or q is cubefree, then there exists $1 \leq j \leq 2r$ with $A_j \neq 0$ such that*

$$\left| \sum_{x \bmod q} \chi(f_1(x) \cdot f_2^{(\phi(q)-1)}(x)) \right| \leq (4r)^{\omega(q)} \sqrt{q} (A_j, q),$$

where (A_j, q) is the greatest common divisor of A_j and q. In particular,

$$\left| \sum_{x \bmod q} \chi(f_1(x) \cdot f_2^{(\phi(q)-1)}(x)) \right| \leq \sum_{\substack{j=1 \\ A_j \neq 0}}^{2r} (4r)^{\omega(q)} \sqrt{q} (A_j, q).$$

Definition 2.2. Define

$$s_q(r,B) = \sum_{1\le b_1\le B}\sum_{1\le b_2\le B}\cdots\sum_{1\le b_{2r}\le B}\min\big\{(A_j,q)\colon 1\le j\le 2r,\ A_j\ne 0\big\}.$$

First, observe that from the trivial bound

$$s_q(r,B) \le \sum_{1\le b_1\le B}\sum_{1\le b_2\le B}\cdots\sum_{1\le b_{2r}\le B}\sum_{\substack{1\le j\le 2r\\ A_j\ne 0}}(A_j,q) \le B^{2r}q.$$

Now, the following lemmas give complementary upper bounds for this expression.

Lemma 2.3. *Let $s_q(r,B)$ be defined as in Definition 2.2. Then*

$$s_q(r,B) \le 2r\Big(\frac{\tau(q)}{2}\Big)^{2r-1}B^{2r}.$$

Proof. Using the bound

$$(A_j,q) = \Bigg(\prod_{\substack{1\le i\le 2r\\ i\ne j}}(b_i-b_j),q\Bigg) \le \prod_{\substack{1\le i\le 2r\\ i\ne j}}(b_i-b_j,q),$$

and exchanging the order of summation, we have

(3)

$$\begin{aligned}
s_q(r,B) \le \sum_{1\le j\le 2r}\sum_{\substack{1\le b_1,b_2,\ldots,b_{2r}\le B\\ A_j\ne 0}}(A_j,q) &\le \sum_{1\le j\le 2r}\sum_{\substack{1\le b_1,b_2,\ldots,b_{2r}\le B\\ A_j\ne 0}}\prod_{\substack{1\le i\le 2r\\ i\ne j}}(b_i-b_j,q)\\
&= \sum_{1\le j\le 2r}\sum_{1\le b_j\le B}\sum_{\substack{1\le b_1,\ldots,b_{j-1},b_{j+1},\ldots,b_{2r}\le B\\ b_j\notin\{b_1,\ldots,b_{j-1},b_{j+1},\ldots,b_{2r}\}}}\prod_{\substack{1\le i\le 2r\\ i\ne j}}(b_i-b_j,q)\\
&= \sum_{1\le j\le 2r}\sum_{1\le b_j\le B}\Bigg(\sum_{\substack{1\le b\le B\\ b\ne b_j}}(b-b_j,q)\Bigg)^{2r-1}\\
&= 2r\sum_{1\le a\le B}\Bigg(\sum_{\substack{1\le b\le B\\ b\ne a}}(b-a,q)\Bigg)^{2r-1}.
\end{aligned}$$

Using the identity $\sum_{d|n}\phi(d)=n$, we write the inner sum as

$$\sum_{\substack{1\le b\le B\\ b\ne a}}(b-a,q) = \sum_{\substack{1\le b\le B\\ b\ne a}}\sum_{d|(b-a,q)}\phi(d) = \sum_{\substack{d|q\\ d\le B-1}}\phi(d)\sum_{\substack{1\le b\le B\\ b\ne a\\ d|b-a}}1 \le \sum_{\substack{d|q\\ d\le B}}\phi(d)\frac{\lfloor B\rfloor}{d} = \lfloor B\rfloor\sum_{\substack{d|q\\ d\le B}}\frac{\phi(d)}{d},$$

where since the divisors of q pair up on either side of $\sqrt{q}$ and $B<\sqrt{q}$,

$$\lfloor B\rfloor\sum_{\substack{d|q\\ d\le B}}\frac{\phi(d)}{d} \le \lfloor B\rfloor\sum_{\substack{d|q\\ d\le B}}1 \le \frac{\lfloor B\rfloor}{2}\tau(q). \tag{4}$$

Putting inequalities (3) and (4) together,

$$
\begin{aligned}
s_q(r,B) &\le 2r \sum_{1\le a\le B}\left(\sum_{\substack{1\le b\le B\\ b\ne a}}(b-a,q)\right)^{2r-1}\\
&\le 2r\sum_{1\le a\le B}\left(\frac{\lfloor B\rfloor}{2}\tau(q)\right)^{2r-1} = 2r\left(\frac{\tau(q)}{2}\right)^{2r-1}\lfloor B\rfloor^{2r},
\end{aligned}
$$

as stated in the lemma. □

We bound the sum $s_q(r,B)$ below without using the initial inequality on the gcd in Lemma 2.3.

Lemma 2.4. *Let $s_q(r,B)$ be defined as in Definition 2.2. Then*

$$s_q(r,B)\le 2rB^{2r}\tau_{2r}(q).$$

Proof. Exchanging the sum as before and using the identity $\sum_{d|n}\phi(d)=n$,

$$
\begin{aligned}
s_q(r,B) \le \sum_{1\le j\le 2r}\sum_{\substack{1\le b_1,b_2,\ldots,b_{2r}\le B\\ A_j\ne 0}}(A_j,q) &= \sum_{1\le j\le 2r}\sum_{\substack{1\le b_1,b_2,\ldots,b_{2r}\le B\\ A_j\ne 0}}\sum_{d|(A_j,q)}\phi(d)\\
&\le \sum_{1\le j\le 2r}\sum_{\substack{d|q\\ d\le B^{2r-1}}}\phi(d)\sum_{\substack{1\le b_1,b_2,\ldots,b_{2r}\le B\\ A_j\ne 0\\ d|A_j}}1.
\end{aligned}
$$

Let $d_1\cdots d_{2r}$ be an ordered factorization of d with $d_j=1$, then $d\mid A_j$ if $d_i\mid(b_i-b_j)$ for each $1\le i\le 2r$. Every inner summand arises in this way for at least one such ordered factorization, therefore

$$\sum_{1\le j\le 2r}\sum_{\substack{d|q\\ d\le B^{2r-1}}}\phi(d)\sum_{\substack{1\le b_1,b_2,\ldots,b_{2r}\le B\\ A_j\ne 0\\ d|A_j}}1 \le \sum_{1\le j\le 2r}\sum_{\substack{d|q\\ d\le B^{2r-1}}}\phi(d)\sum_{1\le b_j\le B}\sum_{\substack{d_1\cdots d_{2r}=d\\ d_j=1}}\sum_{\substack{1\le b_1\le B\\ b_1\ne b_j\\ d_1|(b_1-b_j)}}\cdots\sum_{\substack{1\le b_{2r}\le B\\ b_{2r}\ne b_j\\ d_{2r}|(b_{2r}-b_j)}}1.$$

Note that there are $2r-1$ inner summands with b_j fixed. Bounding each inner summand by $\frac{\lfloor B\rfloor}{d_i}$,

$$
\begin{aligned}
s_q(r,B) &\le \sum_{1\le j\le 2r}\sum_{\substack{d|q\\ d\le B^{2r-1}}}\phi(d)\sum_{1\le b_j\le B}\sum_{\substack{d_1\cdots d_{2r}=d\\ d_j=1}}\sum_{\substack{1\le b_1\le B\\ b_1\ne b_j\\ d_1|(b_1-b_j)}}\cdots\sum_{\substack{1\le b_{2r}\le B\\ b_{2r}\ne b_j\\ d_{2r}|(b_{2r}-b_j)}}1\\
&\le \sum_{1\le j\le 2r}\sum_{\substack{d|q\\ d\le B^{2r-1}}}\phi(d)\sum_{1\le b_j\le B}\sum_{\substack{d_1\cdots d_{2r}=d\\ d_j=1}}\prod_{\substack{1\le i\le 2r\\ i\ne j}}\frac{\lfloor B\rfloor}{d_i} \qquad (5)\\
&= \lfloor B\rfloor^{2r}\sum_{1\le j\le 2r}\sum_{\substack{d|q\\ d\le B^{2r-1}}}\frac{\phi(d)}{d}\sum_{\substack{d_1\cdots d_{2r}=d\\ d_j=1}}1 = 2r\lfloor B\rfloor^{2r}\sum_{\substack{d|q\\ d\le B^{2r-1}}}\frac{\phi(d)}{d}\tau_{2r-1}(d).
\end{aligned}
$$

Since

$$\sum_{d|q} \frac{\phi(d)}{d} \tau_{2r-1}(d) = \prod_{p^k \| q} \left(1 + \frac{p-1}{p} \left(\tau_{2r}(p^k) - 1\right)\right) \leq \tau_{2r}(q),$$

combining with inequality (5), we obtain

$$s_q(r, B) \leq \sum_{1 \leq j \leq 2r} \sum_{\substack{1 \leq b_1, b_2, \ldots, b_{2r} \leq B \\ A_j \neq 0}} (A_j, q) \leq 2r \lfloor B \rfloor^{2r} \tau_{2r}(q),$$

as stated above. □

Using the Lemmas above, we prove the explicit Weil-type inequality.

Proof of Theorem 1.3. As done in the proof of Theorem 1.1 in [17], letting $f_1(x)$ and $f_2(x)$ be defined as in Lemma 2.1, we have

$$\sum_{x=1}^{q} \left| \sum_{b \in \mathcal{B}} \chi(x+b) \right|^{2r} = \sum_{1 \leq b_1, b_2, \ldots, b_{2r} \leq B} \sum_{x \bmod q} \chi(f_1(x) f_2^{(\phi(q)-1)}(x)).$$

Let $(b_1, b_2, \ldots, b_{2r})$ be a good tuple if at least $r+1$ of them are distinct. Let $\mathcal{B}_1$ be the set of good tuples and $\mathcal{B}_2$ be the rest. By Lemma 2.1,

$$\sum_{(b_1, \ldots, b_{2r}) \in \mathcal{B}_1} \sum_{x \bmod q} \chi(f_1(x) f_2^{(\phi(q)-1)}) \leq (4r)^{\omega(q)} \sqrt{q} s_q(r, B).$$

Now using Lemmas 2.3 and 2.4, we obtain the upper bound

$$\sum_{(b_1, \ldots, b_{2r}) \in \mathcal{B}_1} \sum_{x \bmod q} \chi(f_1(x) f_2^{(\phi(q)-1)}) \leq 2r (4r)^{\omega(q)} m_r(q) \sqrt{q} \lfloor B \rfloor^{2r}, \tag{6}$$

where $m_r(q)$ is given in (2).

For the other tuples $\mathcal{B}_2$, we use the trivial bound,

$$\sum_{x \bmod q} \chi(f_1(x) f_2^{(\phi(q)-1)}(x)) \leq q.$$

If $(b_1, b_2, \ldots, b_{2r}) \in \mathcal{B}_2$, then it has at most r distinct values. There are $\binom{\lfloor B \rfloor}{r}$ ways of choosing the r values and each b_i has r choices to make, therefore,

$$\sum_{(b_1, b_2, \ldots, b_{2r}) \in \mathcal{B}_2} \sum_{x \bmod q} \chi(f_1(x) f_2^{(\phi(q)-1)}(x)) \leq r^{2r} \binom{\lfloor B \rfloor}{r} q \leq \frac{r^{2r} \lfloor B \rfloor^r}{r!} q. \tag{7}$$

Combining the bounds in (6) and (7), we get the desired upper bound in the Weil-type inequality. □

3. General setup

Let $r \geq 2, N$ be positive integers, and let q be a positive integer that is cubefree when $r \geq 3$ for the remainder of the paper. Given a primitive Dirichlet character $\chi \pmod q$ and integers M and $N \geq 1$, we consider the character sum

$$S_\chi(M, N) = \sum_{M < n \leq M+N} \chi(n).$$

Given an integer $r \geq 2$, we use induction to prove a bound of the form

$$|S_\chi(M,N)| \leq E_{q,r}(N), \quad \text{where } E_{q,r}(N) = CN^{1-1/r} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} T(q), \tag{8}$$

where $C \geq 1$ is a constant to be determined later, $m_r(q)$ is defined in (1), and

$$T(q) = \begin{cases} \left((4r)^{\omega(q)} m_r(q)\right)^{\left(\frac{1}{2r} - \frac{1}{2r^2}\right)} \left(\frac{q}{\phi(q)}\right)^{1/r} & \text{under the conditions of Theorem 1.1,} \\ \left((4r)^{\omega(q)} m_r(q)\right)^{\left(\frac{1}{2r}\right)} \left(\frac{q}{\phi(q)}\right)^{1/r} & \text{under the conditions of Theorem 1.2.} \end{cases} \tag{9}$$

We take $1 \leq N \leq q^{\frac{1}{4}+\frac{1}{4r}}$ as the base case. In this case, the inequality (8) follows from the trivial bound $|S_\chi(M,N)| \leq N$, since

$$\begin{aligned} & N \leq CN^{1-1/r} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} T(q) \\ \implies & N^{1/r} \leq Cq^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} T(q) \\ \implies & N \leq q^{\frac{r+1}{4r}} \left(C (\log q)^{\frac{1}{2r}} T(q)\right)^r, \end{aligned}$$

when $N \leq q^{\frac{1}{4}+\frac{1}{4r}}$ (because $C(\log q)^{\frac{1}{2r}} T(q) \geq 1$).
Therefore, in the induction step, we assume

$$\text{for } N > q^{\frac{1}{4}+\frac{1}{4r}}, \text{ the bound in (8) holds for all smaller positive integers.} \tag{10}$$

We use the following notation for the remainder of this paper.

Definition 3.1. For some real numbers $A, B \geq 2$, let

- $\mathcal{A} = \{1 \leq a \leq A \colon a \in \mathbb{Z},\ (a,q) = 1\}$,
- $\mathcal{B} = \{1 \leq b \leq B \colon b \in \mathbb{Z}\}$,
- $v_\mathcal{A}(x) = \#\{a \in \mathcal{A},\ n \in (M, M+N] \colon \bar{a}n \equiv x \pmod{q}\}$, where $\bar{a}$ denotes the multiplicative inverse of a modulo q.

We prove the following lemma using the notations in (8), (9), and Definition 3.1.

Lemma 3.2. *For positive integers $r \geq 2$ and q, where q is cubefree for $r \geq 3$, we have, under the inductive hypothesis in (10),*

$$\begin{aligned} |S_\chi(M,N)| \leq{} & \frac{1}{\#\mathcal{A} \cdot \#\mathcal{B}} \left(\sum_{x=1}^{q} v_\mathcal{A}(x)\right)^{1-\frac{1}{r}} \left(\sum_{x=1}^{q} v_\mathcal{A}^2(x)\right)^{\frac{1}{2r}} \left(\sum_{x=1}^{q} \left|\sum_{b \in \mathcal{B}} \chi(x+b)\right|^{2r}\right)^{\frac{1}{2r}} \\ & + \frac{2C}{2-1/r} A^{1-1/r} \frac{(\lfloor B \rfloor + 1)^{2-1/r}}{\#\mathcal{B}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} T(q), \end{aligned}$$

for some constant $C \geq 1$.

Proof. Under the inductive hypothesis in (10), we have, by equation (28) in [16],

$$|S_\chi(M,N)| \leq \frac{1}{\#\mathcal{A} \cdot \#\mathcal{B}} \sum_{x=1}^{q} v_\mathcal{A}(x) \left|\sum_{1 \leq b \leq B} \chi(x+b)\right| + \frac{1}{\#\mathcal{A} \cdot \#\mathcal{B}} \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} 2E_{q,r}(ab), \tag{11}$$

where $E_{q,r}(ab)$ is defined as in (8).

Since $E_{q,r}(N)$ is an increasing function of N,

$$\frac{1}{\#\mathcal{A}\cdot\#\mathcal{B}}\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}2E_{q,r}(ab)\le\frac{1}{\#\mathcal{A}\cdot\#\mathcal{B}}\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}2E_{q,r}(Ab)=\frac{2}{\#\mathcal{B}}\sum_{b\in\mathcal{B}}E_{q,r}(Ab),$$

and by definition,

$$\begin{aligned}\frac{2}{\#\mathcal{B}}\sum_{b\in\mathcal{B}}E_{q,r}(Ab)&=\frac{2}{\#\mathcal{B}}\sum_{b\in\mathcal{B}}C(Ab)^{1-1/r}q^{\frac{r+1}{4r^2}}(\log q)^{\frac{1}{2r}}T(q)\\&=\frac{2C}{\#\mathcal{B}}A^{1-1/r}q^{\frac{r+1}{4r^2}}(\log q)^{\frac{1}{2r}}T(q)\sum_{b\in\mathcal{B}}b^{1-1/r}.\end{aligned}$$

Since

$$\sum_{b\in\mathcal{B}}b^{1-1/r}\le\int_1^{\lfloor B\rfloor+1}t^{1-1/r}\,\mathrm{d}t=\frac{(\lfloor B\rfloor+1)^{2-1/r}-1}{2-1/r}<\frac{(\lfloor B\rfloor+1)^{2-1/r}}{2-1/r},$$

we conclude that

$$\frac{1}{\#\mathcal{A}\cdot\#\mathcal{B}}\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}2E_{q,r}(ab)<\frac{2C}{2-1/r}A^{1-1/r}\frac{(\lfloor B\rfloor+1)^{2-1/r}}{\#\mathcal{B}}q^{\frac{r+1}{4r^2}}(\log q)^{\frac{1}{2r}}T(q).\tag{12}$$

Therefore, combining (11), (12), and applying Hölder's inequality, we get

$$\begin{aligned}|S_\chi(M,N)|\le{}&\frac{1}{\#\mathcal{A}\cdot\#\mathcal{B}}\left(\sum_{x=1}^{q}v_{\mathcal{A}}(x)\right)^{1-\frac{1}{r}}\left(\sum_{x=1}^{q}v_{\mathcal{A}}^2(x)\right)^{\frac{1}{2r}}\left(\sum_{x=1}^{q}\left|\sum_{b\in\mathcal{B}}\chi(x+b)\right|^{2r}\right)^{\frac{1}{2r}}\\&+\frac{2C}{2-1/r}A^{1-1/r}\frac{(\lfloor B\rfloor+1)^{2-1/r}}{\#\mathcal{B}}q^{\frac{r+1}{4r^2}}(\log q)^{\frac{1}{2r}}T(q),\end{aligned}$$

as stated in the lemma. □

In the next section, we handle the two sums involving $v_{\mathcal{A}}$.

4. The $v_{\mathcal{A}}$ sums

Recall from Definition 3.1 that for any real number $A\ge 2$, $\mathcal{A}=\{1\le a\le A\colon a\in\mathbb{Z},\ (a,q)=1\}$ and $v_{\mathcal{A}}(x)=\#\{a\in\mathcal{A},\ n\in(M,M+N]\colon \bar{a}n\equiv x \pmod q\}$, where $\bar{a}$ denotes the multiplicative inverse of a modulo q. We first prove the following lemma.

Lemma 4.1. *Let $\mathcal{A}$ and $v_{\mathcal{A}}(x)$ be defined as in Definition 3.1. Then*

$$\sum_{x \pmod q}v_{\mathcal{A}}(x)=\#\mathcal{A}\cdot N.$$

Proof. By definition,

$$\sum_{x \pmod q}v_{\mathcal{A}}(x)=\sum_{x \pmod q}\#\big\{a\in\mathcal{A},\ n\in(M,M+N]\colon \bar{a}n\equiv x \pmod q\big\}.$$

Summing over all x satisfying $x\equiv\bar{a}n \pmod q$, each ordered pair (a,n) with $a\in\mathcal{A}$ and $n\in(M,M+N]$ is counted exactly once. Therefore, we have the equality stated in the lemma. □

Lemma 4.2. *Let $\mathcal{A}$ and $v_{\mathcal{A}}(x)$ be defined as in Definition 3.1. Then*

$$\sum_{x \pmod q} v_{\mathcal{A}}(x)^2 \leq \sum_{a_1 \in \mathcal{A}} \sum_{a_2 \in \mathcal{A}} \left(1 + \frac{N(a_1, a_2)}{\max\{a_1, a_2\}}\right).$$

Proof. This is the first displayed inequality under Equation (14) in [11]. □

Lemma 4.3. *For all integers d and $N \geq 2$,*

$$\sum_{d \leq N} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O^*\left(\frac{1}{N-1}\right) \quad \text{and} \quad \sum_{d \leq N} \frac{\mu(d) \log d}{d^2} = \frac{36\zeta'(2)}{\pi^4} + O^*\left(\frac{\log(N-1)+1}{N-1}\right).$$

Proof. These equalities follow from evaluations of the corresponding infinite series and the usual bounding of the tail sum by the integral of a decreasing function. □

Lemma 4.4. *For all integers n and $N \geq 2$, we have*

$$\sum_{n \leq N} \frac{\phi(n)}{n^2} \leq \frac{6}{\pi^2} \log N + \delta + \frac{2 \log N + 2}{N-1},$$

where $\delta = \dfrac{6}{\pi^2} - \dfrac{36\zeta'(2)}{\pi^4} \approx 0.954422$.

Proof. Since $\phi(n)/n = \sum_{d|n} \mu(d)/d$,

$$\begin{aligned}
\sum_{n \leq N} \frac{\phi(n)}{n^2} = \sum_{n \leq N} \frac{1}{n} \sum_{d|n} \frac{\mu(d)}{d} &= \sum_{d \leq N} \frac{\mu(d)}{d} \sum_{\substack{n \leq N \\ d|n}} \frac{1}{n} \\
&= \sum_{d \leq N} \frac{\mu(d)}{d^2} \sum_{m \leq N/d} \frac{1}{m} \\
&= \sum_{d \leq N} \frac{\mu(d)}{d^2} \left(\log \frac{N}{d} + O^*(1)\right) \\
&= (\log N + O^*(1)) \sum_{d \leq N} \frac{\mu(d)}{d^2} - \sum_{d \leq N} \frac{\mu(d) \log d}{d^2}.
\end{aligned}$$

By Lemma 4.3,

$$\begin{aligned}
\sum_{n \leq N} \frac{\phi(n)}{n^2} &= (\log N + O^*(1)) \left(\frac{6}{\pi^2} + O^*\left(\frac{1}{N-1}\right)\right) - \left(\frac{36\zeta'(2)}{\pi^4} + O^*\left(\frac{\log(N-1)+1}{N-1}\right)\right) \\
&\leq (\log N + 1)\left(\frac{6}{\pi^2} + \frac{1}{N-1}\right) - \frac{36\zeta'(2)}{\pi^4} + \frac{\log N + 1}{N-1}
\end{aligned}$$

which is the desired inequality. □

The following proposition asymptotically improves the inequality used in [11] by a factor of $6/\pi^2$.

Proposition 4.5. *Let $A \geq 2$ be a real number, $N \geq 2$ be an integer, and $\mathcal{A}, v_{\mathcal{A}}(x)$ be defined as in Definition 3.1. Then*

$$\sum_{x \pmod q} v_{\mathcal{A}}(x)^2 \leq \#\mathcal{A}^2 + 2AN\left(\frac{6}{\pi^2}\log A + \delta + \frac{2\log A + 2}{A-1}\right),$$

where δ is defined in Lemma 4.4.

Proof. Starting from Lemma 4.2,

$$\sum_{x \pmod q} v_{\mathcal{A}}(x)^2 \leq \sum_{a_1\in\mathcal{A}}\sum_{a_2\in\mathcal{A}}\left(1 + \frac{N(a_1,a_2)}{\max\{a_1,a_2\}}\right) = (\#\mathcal{A})^2 + N\sum_{a_1\in\mathcal{A}}\sum_{a_2\in\mathcal{A}}\frac{(a_1,a_2)}{\max\{a_1,a_2\}}$$
$$\leq (\#\mathcal{A})^2 + N\sum_{1\leq a_1\leq A}\sum_{1\leq a_2\leq A}\frac{(a_1,a_2)}{\max\{a_1,a_2\}}.$$

In the remaining sum we use our beloved trick $g = \sum_{d|g}\phi(d)$ to obtain

$$\begin{aligned}
\sum_{1\leq a_1\leq A}\sum_{1\leq a_2\leq A}\frac{(a_1,a_2)}{\max\{a_1,a_2\}} &= \sum_{a_1\in\mathcal{A}}\sum_{a_2\in\mathcal{A}}\frac{1}{\max\{a_1,a_2\}}\sum_{d|(a_1,a_2)}\phi(d)\\
&= \sum_{d\leq A}\phi(d)\sum_{\substack{1\leq a_1\leq A\\ d|a_1}}\sum_{\substack{1\leq a_2\leq A\\ d|a_2}}\frac{1}{\max\{a_1,a_2\}}\\
&= \sum_{d\leq A}\frac{\phi(d)}{d}\sum_{1\leq b_1\leq A/d}\sum_{1\leq b_2\leq A/d}\frac{1}{\max\{b_1,b_2\}}\\
&\leq 2\sum_{d\leq A}\frac{\phi(d)}{d}\sum_{1\leq b_1\leq A/d}\sum_{1\leq b_2\leq b_1}\frac{1}{b_1}\\
&= 2\sum_{d\leq A}\frac{\phi(d)}{d}\sum_{1\leq b_1\leq A/d}1 \leq 2A\sum_{d\leq A}\frac{\phi(d)}{d^2}.
\end{aligned}$$

Now the proposition follows from Lemma 4.4. ∎

5. Important bounds

As we can see in Lemma 3.2, we need to make some choices for A and B to get a Burgess inequality. The following lemmas guarantee that the A and B we pick later satisfy certain properties that are required in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 5.1. *For $r \geq 2$, let $a(r) = 2(3.0758r + 1.38402\log(4r) - 1.5379)\log 2$. If $q \geq e^{e^{a(r)}}$, then*

$$B_1(r,q) \geq 2^{2-2/r}r^2\left(\frac{1-\frac{1}{r}}{r!}\right)^{\frac{1}{r}},$$

where

$$B_1(r,q) = r^2 q^{\frac{1}{2r}}\left(\frac{r-1}{r!2r(4r)^{\omega(q)}m_r(q)}\right)^{1/r}. \tag{13}$$

Proof. Since $m_r(q) \le \big(\tau(q)/2\big)^{2r-1}$ by definition (1) we have

$$B_1(r,q) \ge r^2 q^{\frac{1}{2r}} \left(\frac{2^{2r-1}(r-1)}{2r(r!)(4r)^{\omega(q)}(\tau(q))^{2r-1}} \right)^{1/r}.$$

We have $2^{\omega(q)} \le 2^{1.38402 \log q/\log\log q}$ by [14] and $\tau(q) \le 2^{1.5379 \log q/\log\log q}$ [14, Theorem 1], therefore

$$(4r)^{\omega(q)} \le q^{\frac{1.38402 \log 2 \log (4r)}{\log\log q}},$$

and

$$B_1(r,q) \ge 2^{2-2/r} r^2 \left(\frac{1-\frac{1}{r}}{r!} \right)^{\frac{1}{r}} q^{\frac{1}{2r} - \frac{1.38402(\log 2)\log(4r)}{r \log\log q} - \frac{1.5379(2r-1)\log 2}{r\log\log q}}.$$

When $q \ge e^{e^{a(r)}}$, we have

$$\frac{1}{2r} - \frac{1.38402(\log 2)\log(4r)}{r\log\log q} - \frac{1.5379(2r-1)\log 2}{r\log\log q} \ge 0,$$

so

$$B_1(r,q) \ge 2^{2-2/r} r^2 \left(\frac{1-\frac{1}{r}}{r!} \right)^{\frac{1}{r}}$$

as desired. □

Lemma 5.2. *Let $r \ge 2$. If $q \ge 2^{4r-2}$, then*

$$B_2(r,q) \ge 2^{2-2/r} r^2 \left(\frac{1-\frac{1}{r}}{r!} \right)^{\frac{1}{r}},$$

where

$$B_2(r,q) = r^2 q^{\frac{1}{2r}} \left(\frac{r-1}{r!2r} \right)^{1/r}. \tag{14}$$

Proof. Indeed, for $q \ge 2^{4r-2}$,

$$r^2 q^{\frac{1}{2r}} \left(\frac{r-1}{r!2r} \right)^{1/r} \ge 2^{2-2/r} r^2 \left(\frac{1-\frac{1}{r}}{r!} \right)^{\frac{1}{r}}.$$

□

Lemma 5.3. *For integers $r, q \ge 2$, we have*

$$B_i(r,q) \le erq^{\frac{1}{2r}},$$

for $i = 1, 2$ with the $B_i(r,q)$ defined as in (13) and (14).

Proof. Using $\frac{r-1}{2r(4r)^{\omega(q)} m_r(q)} \le 1$ and $r! \ge \left(\frac{r}{e}\right)^r$,[1] we get

$$\frac{B_1(r,q)}{r^2 q^{\frac{1}{2r}}} = \left(\frac{r-1}{r!(2r)(4r)^{\omega(q)} m_r(q)} \right)^{\frac{1}{r}} \le \frac{e}{r}.$$

---

[1] Since $e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} \ge \sum_{k=n}^{n} \frac{n^k}{k!} = \frac{n^n}{n!}$, we have $n! \ge \frac{n^n}{e^n}$.

Using $r! \geq \left(\frac{r}{e}\right)^r$, we get

$$\frac{B_2(q,r)}{q^{\frac{1}{2r}} r^2} = \left(\frac{r-1}{r!(2r)}\right)^{\frac{1}{r}} \leq \frac{e}{r}.$$

□

To prove our main theorems, we also need some lower bounds on A. In particular, we need $A \geq 31$, and $A \geq 2^{\omega(q)} q/\phi(q)$. The following lemmas cover this for two different choices of A.

Lemma 5.4. *Let $N \geq q^{\frac{1}{4}+\frac{1}{4r}}$ and $r \geq 2$ be positive integers. Suppose*

$$A_i(r,q) = \frac{\kappa N}{B_i(r,q)},$$

for $i = 1, 2$ where $B_1(r,q)$ is defined as in (13), $B_2(r,q)$ as in (14), and let

$$\kappa = \left(\frac{(B_i(r,q)-1)B_i(r,q)^{1-1/r}}{2r(B_i(r,q)+1)^{2-1/r}}\right)^{r/(r-1)}. \tag{15}$$

Let $t_1(r) = e^{e^{a(r)}}$, where $a(r)$ is defined as in Lemma 5.1 and $t_2(r) = 2^{4r-2}$. Then, for $q \geq \max\{10^{1145}, (10r)^{18}, t_i(r)\}$, we have

$$A_i(r,q) \geq \max\left\{31, \frac{2^{\omega(q)} q}{\phi(q)}\right\}.$$

Proof. Since the proof is the same for $i \in \{1, 2\}$, let $B = B_i(r,q)$ and $A = A_i(r,q)$ for brevity. By Lemmas 5.1 and 5.2, we have $B \geq 4$ for $q \geq t_i(r)$. Therefore, for $q \geq t_i(r)$,

$$\kappa \geq \frac{36}{125}\left(\frac{1}{2r}\right)^{r/(r-1)}.$$

Then, by Lemma 5.3,

$$A = \frac{\kappa N}{B} \geq \frac{\kappa q^{\frac{1}{4}+\frac{1}{4r}}}{erq^{\frac{1}{2r}}} \geq \frac{36}{125e}\frac{q^{\frac{1}{4}-\frac{1}{4r}}}{r(2r)^{r/(r-1)}}.$$

The right-hand side exceeds 31 once

$$q \geq \left(31\left(\frac{125e}{36}\right) r(2r)^{r/(r-1)}\right)^{4r/(r-1)},$$

and one can check that this right-hand side is at most $\max\{10^{32}, (25r)^8\}$ for all real $r \geq 2$.

Moreover, we have $2^{\omega(q)} \leq 2^{1.38402 \log q/\log\log q}$ by [14] and

$$\frac{q}{\phi(q)} < e^{\gamma} \log\log q + \frac{2.50637}{\log\log q} < 2\log\log q$$

by [15, Theorem 15]. Thus it suffices to show that

$$\frac{q^{\frac{1}{4}-\frac{1}{4r}}}{r(2r)^{r/(r-1)}} \geq 2^{1.38402 \log q/\log\log q} \cdot 2\log\log q \cdot \frac{125e}{36}$$

or equivalently

$$q \geq \left(r(2r)^{r/(r-1)}\right)^{(\frac{1}{4}-\frac{1}{4r})^{-1}} \left(2^{1.38402 \log q/\log\log q} \cdot 2\log\log q \cdot\right)^{(\frac{1}{4}-\frac{1}{4r})^{-1}} \left(\frac{125e}{36}\right)^{(\frac{1}{4}-\frac{1}{4r})^{-1}}.$$

When $2 \leq r \leq 9$ we check computationally that this inequality holds for $q \geq 10^{1145}$. For $r \geq 10$, we use the checkable bound

$$\left(r(2r)^{r/(r-1)}\right)^{(\frac{1}{4}-\frac{1}{4r})^{-1}} \leq 741r^8$$

together with the trivial $(\frac{1}{4}-\frac{1}{4r})^{-1} \leq \frac{40}{9}$ to see that it suffices for

$$q \geq 741r^8\left(2^{1.38402 \log q/\log\log q} \cdot 2\log\log q\right)^{40/9} \left(\frac{125e}{36}\right)^{40/9}.$$

We can computationally show the inequality

$$2^{1.38402 \log q/\log\log q} \cdot 2\log\log q \leq q^{1/8}$$

for $q \geq 10^{1008}$, and so it suffices to have

$$q \geq 741r^8\left(q^{1/8}\right)^{40/9} \left(\frac{125e}{36}\right)^{40/9},$$

which holds when $q \geq (10r)^{18} \geq (741r^8)^{9/4}\left(\frac{125e}{36}\right)^{10}$. $\square$

Lemma 5.5. *Let $r, q \geq 2$ and $N \geq q^{\frac{1}{4}+\frac{1}{4r}}$ be integers and let $i \in \{1,2\}$. Let $A = A_i(r,q)$, $B = B_i(r,q)$, κ be defined as in (15), and $t_i(r)$ be defined as in Lemma 5.4. Let*

$$\alpha = \left[\frac{\left(\kappa\left(\frac{\phi(q)}{q}\right) + \frac{B2^{\omega(q)-1}}{N}\right)^2}{B} + 2\kappa\left(\frac{6}{\pi^2}\log(A) + \delta + \frac{2\log(A)+2}{A-1}\right)\right],$$

where $\delta = \frac{6}{\pi^2} - \frac{36\zeta'(2)}{\pi^4} \approx 0.954422$. Then, for $q \geq \max\{10^{1145}, (10r)^{18}, t_i(q)\}$,

$$\alpha \leq \frac{4}{3}\kappa \log q.$$

Proof. Since $q \geq \max\{10^{1145}, (10r)^{18}, t_i(q)\}$, we have $A \geq \frac{2^{\omega(q)}q}{\phi(q)}$ by Lemma 5.4. Using $AB = \kappa N$, we obtain

$$\frac{\left(\kappa\phi^* + \frac{B2^{\omega(q)-1}}{N}\right)^2}{B} = \frac{A\left(\kappa\phi^* + \frac{\kappa 2^{\omega(q)-1}}{A}\right)^2}{\kappa N} = \frac{\kappa A\left(\phi^* + \frac{2^{\omega(q)-1}}{A}\right)^2}{N}$$
$$= \frac{\kappa(A\phi^* + 2^{\omega(q)-1})^2}{AN} \leq \frac{\kappa\left(\frac{3}{2}A\phi^*\right)^2}{AN} \leq \frac{9\kappa A}{4N} \leq \frac{9\kappa}{4}.$$

We also have $A \geq 31$, which implies

$$\delta + \frac{2\log(A)+2}{A-1} \leq \frac{3}{8}\log A.$$

We may assume $N \le q^{\frac{2}{3}}$ because otherwise the Pólya–Vinogradov inequality produces a better bound. Thus $A \le q^{\frac{2}{3}}$, and for $q \ge 10^{58}$,

$$\alpha \le \frac{9\kappa}{4} + 2\kappa \log A \left(\frac{6}{\pi^2} + \frac{3}{8}\right) \le \frac{9\kappa}{4} + \frac{4}{3}\kappa \log q \left(\frac{3}{8} + \frac{6}{\pi^2}\right)$$
$$= \frac{4}{3}\kappa \log q \left(\frac{27}{16 \log q} + \frac{3}{8} + \frac{6}{\pi^2}\right) < \frac{4}{3}\kappa \log q. \qquad \square$$

6. Proof of Main Theorems

In this section, we prove Theorems 1.1 and 1.2. Recall that we let $r \ge 2$, q be positive integers and $i \in \{1, 2\}$. We let $i = 1$ be the case for Theorem 1.1, and $i = 2$ for Theorem 1.2. Let

$$T_1(q) = \left((4r)^{\omega(q)} m_r(q)\right)^{\left(\frac{1}{2r} - \frac{1}{2r^2}\right)} \left(\frac{q}{\phi(q)}\right)^{1/r}, \tag{16}$$

and

$$T_2(q) = \left((4r)^{\omega(q)} m_r(q)\right)^{\left(\frac{1}{2r}\right)} \left(\frac{q}{\phi(q)}\right)^{1/r}. \tag{17}$$

Let $t_i(r)$ be defined as in Lemma 5.4. Our goal is to prove that for $q \ge \max\{10^{1145}, t_i(r)\}$, there exists a constant $C(r)$ depending on r (but not q) such that

$$|S_\chi(M, N)| \le C(r) N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} T_i(q).$$

Lemma 6.1. *For $i \in \{1, 2\}$ and T_i be defined as in (16) and (17). Let s, α, u, and w be defined as in (19), (22), (25), and (26), respectively. Then*

$$|S_\chi(M, N)| \le \left(\frac{B}{B-1}\right) N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{\frac{1}{2r}} \left(u(\alpha w)^{\frac{1}{2r}} + sT_i(q)\right).$$

Proof. Fix $i \in \{1, 2\}$ and let $q \ge \max\{10^{1145}, t_i(r)\}$. From Lemma 3.2, once we choose A and B as described in Definition 3.1, replacing $T(q)$ with $T_i(q)$ and $C(r)$ with C, we have

$$|S_\chi(M, N)| \le \frac{1}{\#\mathcal{A} \cdot \#\mathcal{B}} \left(\sum_{x=1}^{q} v_{\mathcal{A}}(x)\right)^{1-\frac{1}{r}} \left(\sum_{x=1}^{q} v_{\mathcal{A}}^2(x)\right)^{\frac{1}{2r}} \left(\sum_{x=1}^{q} \left|\sum_{b \in \mathcal{B}} \chi(x+b)\right|^{2r}\right)^{\frac{1}{2r}}$$
$$+ \frac{2C}{2 - 1/r} A^{1-1/r} \frac{(\lfloor B \rfloor + 1)^{2-1/r}}{\#\mathcal{B}} q^{\frac{r+1}{4r^2}} (\log q)^{\frac{1}{2r}} T_i(q).$$

Let $\phi^* = \phi(q)/q$, $B = B_i(r, q)$,

$$\kappa = \left(\frac{(B-1)B^{1-\frac{1}{r}}}{2r(B+1)^{2-\frac{1}{r}}}\right)^{\frac{r}{r-1}},$$

and

$$A = \frac{\kappa N}{B}.$$

Using (12), Theorem 1.3, Lemma 4.1, and Proposition 4.5 yields [2]

(18)

$$
\begin{aligned}
|S_\chi(M,N)| &\le \frac{1}{\#\mathcal{A}\cdot\lfloor B\rfloor}(\#\mathcal{A}N)^{1-\frac{1}{r}}\left(\#\mathcal{A}^2+2AN\left(\frac{6}{\pi^2}\log(A)+\delta+\frac{2\log(A)+2}{A-1}\right)\right)^{\frac{1}{2r}}\times\\
&\times\left(2r(4r)^{\omega(q)}B^{2r}m_r(q)\sqrt{q}+\frac{r^{2r}}{r!}B^r q\right)^{\frac{1}{2r}}+\frac{2C}{2-\frac{1}{r}}A^{1-\frac{1}{r}}\frac{(B+1)^{2-\frac{1}{r}}}{B}q^{\frac{r+1}{4r^2}}(\log q)^{\frac{1}{2r}}T_i(q)\\
&\le \frac{N^{1-\frac{1}{r}}}{\#\mathcal{A}^{\frac{1}{r}}\cdot(B-1)}\left(\left(\frac{\kappa N}{B}\phi^*+2^{\omega(q)-1}\right)^2+\frac{2\kappa N^2}{B}\left(\frac{6}{\pi^2}\log(A)+\delta+\frac{2\log A+2}{A-1}\right)\right)^{\frac{1}{2r}}\times\\
&\times\left(2r(4r)^{\omega(q)}B^{2r}m_r(q)\sqrt{q}+\frac{r^{2r}}{r!}B^r q\right)^{\frac{1}{2r}}+\frac{2C}{2-\frac{1}{r}}(\kappa N)^{1-\frac{1}{r}}\left(\frac{B+1}{B}\right)^{2-\frac{1}{r}}q^{\frac{r+1}{4r^2}}(\log q)^{\frac{1}{2r}}T_i(q).
\end{aligned}
$$

Let

$$
s=\frac{2C\kappa^{1-1/r}}{2-1/r}\left(\frac{B+1}{B}\right)^{2-1/r}, \tag{19}
$$

$$
P=\left(\frac{\kappa N}{B}\phi^*+2^{\omega(q)-1}\right)^2+\frac{2\kappa N^2}{B}\left(\frac{6}{\pi^2}\log(A)+\delta+\frac{2\log A+2}{A-1}\right),
$$

and

$$
Q=2r(4r)^{\omega(q)}B^{2r}m_r(q)\sqrt{q}+\frac{r^{2r}}{r!}B^r q.
$$

We can write the upper bound for $|S_\chi(M,N)|$ in (18) as

$$
\frac{N^{1-\frac{1}{r}}}{\#\mathcal{A}^{\frac{1}{r}}\cdot(B-1)}P^{\frac{1}{2r}}\times Q^{\frac{1}{2r}}+N^{1-1/r}q^{\frac{r+1}{4r^2}}T_i(q)(\log q)^{\frac{1}{2r}}s. \tag{20}
$$

Factoring out N^2/B from P, we deduce

$$
P=\frac{N^2}{B}\left[\frac{\left(\kappa\phi^*+\frac{B2^{\omega(q)-1}}{N}\right)^2}{B}+2\kappa\left(\frac{6}{\pi^2}\log(A)+\delta+\frac{2\log(A)+2}{A-1}\right)\right] \tag{21}
$$

$$
\implies P^{\frac{1}{2r}}=N^{1/r}\left(\frac{\alpha}{B}\right)^{\frac{1}{2r}},
$$

where for simplicity, we let

$$
\alpha=PB/N^2=\left[\frac{\left(\kappa\phi^*+\frac{B2^{\omega(q)-1}}{N}\right)^2}{B}+2\kappa\left(\frac{6}{\pi^2}\log(A)+\delta+\frac{2\log(A)+2}{A-1}\right)\right]. \tag{22}
$$

Let

$$
\beta=\frac{B}{r^2q^{\frac{1}{2r}}}.
$$

[2] In the first upper bound we actually have $(\lfloor B\rfloor+1)^{2-1/r}/\lfloor B\rfloor$; but the function $(x+1)^{2-1/r}/x$ is increasing for $x\ge 2$, so we can replace $\lfloor B\rfloor$ by B as shown.

Then

$$Q^{\frac{1}{2r}} = q^{\frac{3}{4r}} r^2 \beta^{1/2} \left(\frac{f_i}{r!}\right)^{\frac{1}{2r}}, \tag{23}$$

where $f_1 = r$ and

$$f_2 = (4r)^{\omega(q)} m_r(q)(r-1) + 1.$$

Therefore, writing $B = r^2 q^{\frac{1}{2r}} \beta$, we obtain from Equations (20), (21), and (23) that

$$|S_\chi(M,N)| \leq \frac{N^{1-1/r}}{B-1} \left(\frac{N}{\#\mathcal{A}}\right)^{\frac{1}{r}} q^{\frac{3}{4r}-\frac{1}{4r^2}} r^{2-\frac{1}{r}} \beta^{\frac{1}{2}-\frac{1}{2r}} \left(\frac{\alpha f_i}{r!}\right)^{\frac{1}{2r}} + N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} T_i(q) \log(q)^{\frac{1}{2r}} s. \tag{24}$$

Let

$$u = \frac{(N/\#\mathcal{A})^{1/r}}{q^{\frac{1}{2r^2}}}, \tag{25}$$

and

$$w = \frac{f_i}{r^2 \beta^{r+1} r! \log q}. \tag{26}$$

By multiplying and dividing by B and a few other manipulations we can go from (24) to

$$\begin{aligned} |S_\chi(M,N)| &\leq \left(\frac{B}{B-1}\right) N^{1-1/r} q^{(r+1)/4r^2} T_i(q) (\log q)^{\frac{1}{2r}} \left(\frac{u(\alpha w)^{\frac{1}{2r}}}{T_i(q)} + s\right) \\ &= \left(\frac{B}{B-1}\right) N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{\frac{1}{2r}} \left(u(\alpha w)^{\frac{1}{2r}} + sT_i(q)\right), \end{aligned} \tag{27}$$

as stated in the lemma. □

Using this lemma, we now prove our main theorems separately below.

Proof of Theorem 1.1. Recall that we assume $q \geq \max\{10^{1145}, t_1(r)\}$ for $t_1(r) = e^{e^{a(r)}}$ with $a(r)$ given in the statement of Theorem 1.1 (and Lemma 5.1), and from (16)

$$T_1(q) = \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}-\frac{1}{2r^2}} \left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

By Lemma 6.1 and comparing with (8), it suffices to show that

$$u(\alpha w)^{\frac{1}{2r}} + sT_1(q) \leq \frac{B-1}{B} C T_1(q).$$

Since $10^{1145} \geq (10r)^{18}$ for $r \leq 10^{50}$ while $t_1(r) \geq (10r)^{18}$ for $r \geq 10^{50}$, we apply Lemma 5.4 and conclude that $A\phi^* \geq 2^{\omega(q)}$, hence

$$\frac{N}{\#\mathcal{A}} \leq \frac{N}{A\phi^* - 2^{\omega(q)-1}} = \frac{N}{A\phi^*} + \frac{2^{\omega(q)-1} N}{A\phi^*(A\phi^* - 2^{\omega(q)-1})} \leq \frac{2N}{A\phi^*}.$$

Therefore, (25) satisfies the bound

$$u = \frac{(N/\#\mathcal{A})^{1/r}}{q^{\frac{1}{2r^2}}} \leq \frac{2^{1/r}}{q^{\frac{1}{2r^2}}} \left(\frac{N}{A\phi^*}\right)^{1/r}. \tag{28}$$

From Lemma 5.5, we have

$$\alpha \le \frac{4}{3}\kappa \log q. \tag{29}$$

Using that $f_1 = r$ and

$$\beta = \left(\frac{r-1}{(2r)r!(4r)^{\omega(q)}m_r(q)}\right)^{\frac{1}{r}}, \tag{30}$$

we rewrite (26) as

$$w = \frac{2^{1+1/r}(r\cdot r!)^{1/r}}{(r-1)^{1+1/r}\log q}\left((4r)^{\omega(q)}\, m_r(q)\right)^{1+1/r}. \tag{31}$$

Furthermore, using (30), $B = r^2 q^{\frac{1}{2r}}\beta$, and $AB = \kappa N$, we have

$$\frac{N}{Aq^{\frac{1}{2r}}}\left((4r)^{\omega(q)}m_r(q)\right)^{\frac{1}{r}} = \frac{r^2}{\kappa}\left(\frac{r-1}{r!(2r)}\right)^{\frac{1}{r}}. \tag{32}$$

Therefore, combining (19), (28), (29), (31), and (32), we obtain

$$\begin{aligned}
&u(\alpha w)^{\frac{1}{2r}} + sT_1(q)\\
&\le 2^{\frac{1}{r}}\left(\frac{N}{Aq^{\frac{1}{2r}}\phi^*}\right)^{\frac{1}{r}}\left(\frac{4}{3}\kappa\log q\right)^{\frac{1}{2r}}\left(\frac{2^{1+\frac{1}{r}}(r\cdot r!)^{\frac{1}{r}}r^{\frac{1}{r}}}{(r-1)^{1+\frac{1}{r}}\log q}\right)^{\frac{1}{2r}}\left((4r)^{\omega(q)}m_r(q)\right)^{\frac{1}{2r}+\frac{1}{2r^2}}\\
&+\frac{2C\kappa^{1-\frac{1}{r}}}{2-\frac{1}{r}}\left(\frac{B+1}{B}\right)^{2-\frac{1}{r}}T_1(q)\\
&=\left(2^{\frac{5}{2r}-\frac{1}{2r^2}}3^{-\frac{1}{2r}}(r!)^{-\frac{1}{2r^2}}(r-1)^{\frac{1}{2r^2}-\frac{1}{2r}}r^{\frac{2}{r}-\frac{1}{2r^2}}\kappa^{-\frac{1}{2r}} + \frac{2C\kappa^{1-\frac{1}{r}}}{2-\frac{1}{r}}\left(\frac{B+1}{B}\right)^{2-\frac{1}{r}}\right)T_1(q).
\end{aligned}$$

Therefore,

$$C \ge \frac{2^{\frac{5}{2r}-\frac{1}{2r^2}}3^{-\frac{1}{2r}}(r!)^{-\frac{1}{2r^2}}(r-1)^{\frac{1}{2r^2}-\frac{1}{2r}}r^{\frac{2}{r}-\frac{1}{2r^2}}\kappa^{-\frac{1}{2r}}}{\frac{B-1}{B}-\frac{2\kappa^{1-\frac{1}{r}}}{2-\frac{1}{r}}\left(\frac{B+1}{B}\right)^{2-\frac{1}{r}}}. \tag{33}$$

Now to obtain the constants in Table 1, first, by Lemmas 5.1 and 5.2,

$$B \ge 2^{2-\frac{2}{r}}r^2\left(\frac{1-\frac{1}{r}}{r!}\right)^{\frac{1}{r}}. \tag{34}$$

Then, choosing κ as in (15), we replace B with $2^{2-2/r}r^2\left(\frac{1-\frac{1}{r}}{r!}\right)^{\frac{1}{r}}$ in κ since the right side of (33) is decreasing as B increases for $B \ge 2^{2-2/r}r^2\left(\frac{1-\frac{1}{r}}{r!}\right)^{\frac{1}{r}}$. Computing the resulting expression obtains the $C(r)$ column in Table 1. For the $D(r)$ column in Table 1, we notice that $q\to\infty$ implies $B\to\infty$, therefore

$$\kappa \to \left(\frac{1}{2r}\right)^{\frac{r}{r-1}},$$

and we obtain

$$D(r)=\frac{2^{\frac{5}{2r}-\frac{1}{2r^2}}3^{-\frac{1}{2r}}(r!)^{-\frac{1}{2r^2}}(r-1)^{\frac{1}{2r^2}-\frac{1}{2r}}r^{\frac{2}{r}-\frac{1}{2r^2}}(2r)^{\frac{1}{2r-2}}}{1-\frac{1}{2r-1}}.$$

$\square$

We prove Theorem 1.2 following the same procedure as the proof of Theorem 1.1.

Proof of Theorem 1.2. Recall that we assume $q\geq\max\{10^{1145},t_2(r)\}$ for $t_2(r)=2^{4r-2}$, and from (17)

$$T_2(q)=\left((4r)^{\omega(q)}m_r(q)\right)^{\frac{1}{2r}}\left(\frac{q}{\phi(q)}\right)^{\frac{1}{r}}.$$

By Lemma 6.1, it suffices to show

$$u(\alpha w)^{\frac{1}{2r}}+sT_2(q)\leq\frac{B-1}{B}CT_2(q).$$

Similar to the proof of Theorem 1.1, $10^{1145}\geq(10r)^{18}$ for $r\leq10^{50}$ while $t_2(r)\geq(10r)^{18}$ for $r\geq10^{50}$, thus from (28) and (29),

$$u\leq\frac{2^{1/r}}{q^{\frac{1}{2r^2}}}\left(\frac{N}{A\phi^*}\right)^{1/r},$$

and

$$\alpha\leq\frac{4}{3}\kappa\log q.$$

Furthermore,

$$f_2=(4r)^{\omega(q)}m_r(q)(r-1)+1\leq r(4r)^{\omega(q)}m_r(q), \tag{35}$$

and

$$\beta=\left(\frac{r-1}{(2r)r!}\right)^{\frac{1}{r}}, \tag{36}$$

therefore we bound w in (26) as

$$w\leq\frac{2^{1+1/r}(r\cdot r!)^{1/r}}{(r-1)^{1+1/r}\log q}(4r)^{\omega(q)}m_r(q). \tag{37}$$

Using (36), $B=r^2q^{\frac{1}{2r}}\beta$, and $AB=\kappa N$, we get

$$\frac{N}{Aq^{\frac{1}{2r}}}=\frac{r^2}{\kappa}\left(\frac{r-1}{r!(2r)}\right)^{\frac{1}{r}}. \tag{38}$$

Therefore, combining (19), (28), (29), (37), and (38), we get

$$
\begin{aligned}
&u(\alpha w)^{\frac{1}{2r}} + sT_2(q) \\
&\le 2^{\frac{1}{r}} \left(\frac{N}{Aq^{\frac{1}{2r}}\phi^*}\right)^{\frac{1}{r}} \left(\frac{4}{3}\kappa \log q\right)^{\frac{1}{2r}} \left(\frac{2^{1+\frac{1}{r}}(r\cdot r!)^{\frac{1}{r}} r^{\frac{1}{r}}}{(r-1)^{1+\frac{1}{r}}\log q}\right)^{\frac{1}{2r}} \left((4r)^{\omega(q)} m_r(q)\right)^{\frac{1}{2r}} \\
&+ \frac{2C\kappa^{1-\frac{1}{r}}}{2-\frac{1}{r}} \left(\frac{B+1}{B}\right)^{2-\frac{1}{r}} T_2(q) \\
&= \left(2^{\frac{5}{2r}-\frac{1}{2r^2}} 3^{-\frac{1}{2r}} (r!)^{-\frac{1}{2r^2}} (r-1)^{\frac{1}{2r^2}-\frac{1}{2r}} r^{\frac{2}{r}-\frac{1}{2r^2}} \kappa^{-\frac{1}{2r}} + \frac{2C\kappa^{1-\frac{1}{r}}}{2-\frac{1}{r}} \left(\frac{B+1}{B}\right)^{2-\frac{1}{r}}\right) T_2(q).
\end{aligned}
$$

This produces the same conditions on C and D as in the proof of Theorem 1.1, thus we obtain the constants in Table 1. $\square$

Acknowledgments

The authors thank the PIMS CRG "L-functions in Analytic Number Theory" and the BIRS facilities at the University of British Columbia–Okanagan, who together ran the Inclusive Paths in Explicit Number Theory Summer School where this project started. The third author was supported in part by a Natural Sciences and Engineering Council of Canada Discovery Grant. The fourth and fifth authors also thank FONDOCyT, grant number 2022-1D1-085, for supporting this research.

References

1. Andrew R. Booker, *Quadratic class numbers and character sums*, Math. Comp. **75** (2006), no. 255, 1481–1492 (electronic). MR 2219039 (2008a:11140)
2. D. A. Burgess, *On character sums and L-series*, Proc. London Math. Soc. (3) **12** (1962), 193–206. MR 132733
3. ———, *On character sums and primitive roots*, Proc. London Math. Soc. (3) **12** (1962), 179–192. MR 0132732 (24 #A2569)
4. ———, *A note on the distribution of residues and non-residues*, J. London Math. Soc. **38** (1963), 253–256. MR 0148628 (26 #6135)
5. ———, *On character sums and L-series. II*, Proc. London Math. Soc. (3) **13** (1963), 524–536. MR 0148626 (26 #6133)
6. ———, *The character sum estimate with* $r = 3$, J. London Math. Soc. (2) **33** (1986), no. 2, 219–226. MR 838632
7. Stephen D. Cohen, Tomás Oliveira e Silva, and Tim Trudgian, *On Grosswald's conjecture on primitive roots*, Acta Arith. **172** (2016), no. 3, 263–270. MR 3460815
8. Noam D. Elkies, *Rank of an elliptic curve and 3-rank of a quadratic field via the Burgess bounds*, Res. Number Theory **11** (2025), no. 3, Paper No. 70, 14. MR 4933368
9. Forrest J. Francis, *An investigation into explicit versions of Burgess' bound*, J. Number Theory **228** (2021), 87–107. MR 4271811
10. Henryk Iwaniec and Emmanuel Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR 2061214 (2005h:11005)
11. Niraek Jain-Sharma, Tanmay Khale, and Mengzhen Liu, *Explicit Burgess bound for composite moduli*, Int. J. Number Theory **17** (2021), no. 10, 2207–2219. MR 4322829
12. D. R. Johnston, O. Ramaré, and T. Trudgian, *An explicit upper bound for* $L(1,\chi)$ *when* χ *is quadratic*, Res. Number Theory **9** (2023), no. 4, Paper No. 72, 20. MR 4649452

13. Kevin J. McGown, *Norm-Euclidean cyclic fields of prime degree*, Int. J. Number Theory **8** (2012), no. 1, 227–254. MR 2887892
14. J.-L. Nicolas and G. Robin, *Explicit upper estimates for the number of divisors of n*, Can. Math. Bull. **26** (1983), 485–492 (French).
15. J. Barkley Rosser and Lowell Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94. MR 0137689 (25 #1139)
16. Enrique Treviño, *The Burgess inequality and the least kth power non-residue*, Int. J. Number Theory **11** (2015), no. 5, 1653–1678. MR 3376232
17. ———, *The least k-th power non-residue*, J. Number Theory **149** (2015), 201–224. MR 3296008

(Elchin Hasanalizade) School of Information Technologies and Engineering, ADA University, Baku, Azerbaijan
Email address: `ehasanalizade@ada.edu.az`

(Hua Lin) Department of Mathematics, Northwestern University, Evanston, IL, USA
Email address: `hua.lin@northwestern.edu`

(Greg Martin) Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, BC, Canada V6T 1Z2
Email address: `gerg@math.ubc.ca`

(Andradis Luna Martínez) Escuela de Matemáticas, Universidad Autónoma de Santo Domingo, Santo Domingo, Dominican Republic
Email address: `aluna71@uasd.edu.do`

(Enrique Treviño) Department of Mathematics and Computer Science, Lake Forest College, Lake Forest, IL, USA
Email address: `trevino@lakeforest.edu`