An almost-always measure of the failure of unique factorization

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(joint work with Paul Pollack)

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LAKE FOREST
COLLEGE

Unique Factorization

- Let R be an integral domain (a ring that satisfies that $ab = 0$ implies $a = 0$ or $b = 0$).
- An element $u \in R$ is a unit if $u^{-1} \in R$.
- A nonzero element $r \in R$ is **irreducible** if r is not a unit and $r = ab$ implies a or b is a unit.
- We say R is a **Unique Factorization Domain** if any nonzero, nonunit element $r \in R$ can be factored as a product of irreducibles, and if

$$\pi_1 \pi_2 \cdots \pi_k = r = \rho_1 \rho_2 \cdots \rho_m,$$

with π_i, ρ_j irreducible in R, then

(a) $k = m$, and

(b) for some permutation σ of $\{1, 2, \ldots, k\}$, and some units $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in R$, we have $\pi_{\sigma(i)} = \epsilon_i \rho_i$ for all i.

Examples and non-Examples of Unique Factorization Domains

Examples:

- $\mathbb{Z}$,
- $\mathbb{Z}[x]$,
- The Gaussian integers: $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.

Non-examples:

- $$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\},$$

  because $2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

- $$\mathbb{Z}\left[\frac{1 + \sqrt{-23}}{2}\right] = \left\{\frac{a}{2} + \frac{b}{2}\sqrt{-23} \mid a, b \in \mathbb{Z} \text{ with } a \equiv b \bmod 2\right\},$$

  because $2 \cdot 2 \cdot 2 = 8 = \left(-\frac{3}{2} + \frac{1}{2}\sqrt{-23}\right)\left(-\frac{3}{2} - \frac{1}{2}\sqrt{-23}\right)$.

Algebraic Numbers and Algebraic Integers

Definition

A complex number α is algebraic if there exists a polynomial $p(x)$ with integer coefficients such that $p(\alpha) = 0$.

Definition

A complex number α is an **algebraic integer** if there exists a monic polynomial $p(x)$ with integer coefficients such that $p(\alpha) = 0$.

- $i, \sqrt{2}$ are algebraic integers because they are roots of $x^2 + 1$ and $x^2 - 2$, respectively.
- $\frac{1+\sqrt{-3}}{2}$ is an algebraic integer because it is a root of $x^2 + x + 1$.
- $\frac{\sqrt{-3}}{2}$ is algebraic but not an algebraic integer, because its minimal polynomial is $4x^2 + 3$.

Class Group and Class Number

For any ring of integers $\mathcal{O}_K$ of a number field K, the quotient group of the group of fractional ideals of $\mathcal{O}_K$ over the group of principal fractional ideals of $\mathcal{O}_K$ is called the **Class group** of $\mathcal{O}_K$. The **Class number** is the number of elements of the class group.

Number Field	Ring of Integers	Class Group	Class Number
$\mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}_1$	1
$\mathbb{Q}(i)$	$\mathbb{Z}[i]$	$\mathbb{Z}_1$	1
$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Z}[\sqrt{-5}]$	$\mathbb{Z}_2$	2
$\mathbb{Q}(\sqrt{-23})$	$\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$	$\mathbb{Z}_3$	3

Classification of Unique Factorization Domains

Theorem

A ring of integers $\mathcal{O}_K$ of a number field K is a Unique Factorization Domain if and only if, it has class number 1.

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Half-Factorial Domains

Definition

We say an integral domain R is a **Unique Factorization Domain** if any nonzero, nonunit element $r \in R$ can be factored as a product of irreducibles, and if

$$\pi_1 \pi_2 \cdots \pi_k = r = \rho_1 \rho_2 \cdots \rho_m,$$

with π_i, ρ_j irreducible in R, then

1. $k = m$, and
2. for some permutation σ of $\{1, 2, \ldots, k\}$, and some units $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in R$, we have $\pi_{\sigma(i)} = \epsilon_i \rho_i$ for all i.

Definition

R is a **half-factorial domain** if it satisfies the first condition but not necessarily the second, i.e., if every factorization of an element $\alpha \in R$ into irreducibles has the same length.

Classification of Half-Factorial Domains

Theorem (Carlitz)

A ring of integers $\mathcal{O}_K$ of a number field K is a Half-Factorial Domain if and only if, it has class number ≤ 2.

In particular, while $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain, since its class group is $\mathbb{Z}_2$, then it is a half-factorial domain. So all factorizations of an element in $\mathbb{Z}[\sqrt{-5}]$ have the same length!

Elasticity of an Element

Definition

Let $\alpha \in \mathcal{O}_K$ be nonzero and not a unit. Consider the set of lengths of factorizations of α, i.e.,

$$L = \{\ell \in \mathbb{N} \mid \text{there exist irreducibles } \rho_1, \rho_2, \ldots, \rho_\ell \in \mathcal{O}_K \text{ such that } \rho_1\rho_2\cdots\rho_\ell = \alpha\}.$$

Then the **elasticity** of α is

$$\rho(\alpha) = \frac{\max L}{\min L}.$$

For example, the elasticity of 8 in $\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$ is $3/2$ because the longest way to factor 8 is $2 \cdot 2 \cdot 2$ and the shortest way is

$$\left(\frac{3}{2} + \frac{1}{2}\sqrt{-23}\right)\left(\frac{3}{2} - \frac{1}{2}\sqrt{-23}\right).$$

Elasticity of a Ring of Integers

Definition

The elasticity of a ring of integers $\mathcal{O}_K$ is

$$\rho(\mathcal{O}_K) = \sup_{\alpha} \rho(\alpha).$$

- The elasticity is 1 if and only if the ring of integers is a half-factorial domain.

Davenport Constant

Definition

Let G be a finite abelian group with operation $+$. Then, there exists a smallest positive integer $D(G)$ such that every sequence $g_1, g_2, \ldots, g_{D(G)}$ of elements of G contains a non-empty subsequence whose sum is the identity.

Examples:

- The Davenport constant for a cyclic group of order n is n.
- The Davenport constant for $\mathbb{Z}_2 \times \mathbb{Z}_2$ is 3. That is because
  - If one term is $(0,0)$, we are done.
  - If two terms are equal, their sum is $(0,0)$.
  - Otherwise, the three nonzero elements sum to $(0,0)$.

Theorem of Elasticity

Theorem (Narkiewicz, Steffan, Valenza)

Whenever $\mathcal{O}_K$ is not a unique factorization domain, we have

$$\rho(\mathcal{O}_K) = \frac{1}{2}D(G),$$

where G is the class group of $\mathcal{O}_K$ and $D(G)$ is its Davenport constant.

- From this theorem, it follows that $\mathcal{O}_K$ is a half-factorial domain if and only if its class number is ≤ 2. (Because when the class group has more than 2 elements, its Davenport constant is bigger than 2, so the elasticity is bigger than 1).
- This shows that $\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$ has elasticity $\frac{3}{2}$, because for this ring, the class group is $\mathbb{Z}_3$, so its Davenport constant is 3.

Distribution of values

- Consider $\rho(\alpha)$ as α ranges over the nonzero elements of $\mathcal{O}_K$. Suppose $\mathcal{O}_K$ is not a unique factorization domain. Let G be the class group of $\mathcal{O}_K$.
- We know
$$1 \leq \rho(\alpha) \leq \frac{1}{2} D(G) = \rho(\mathcal{O}_K).$$
- We know that every rational in $[1, \rho(\mathcal{O}_K)]$ is attained by some $\rho(\alpha)$ (Baginski, Chapman, Crutchfield, Grace Kennedy and Wright, 2006).
- What about more statistical properties? Is there a mean value? Is there a typical value?

Theorem (Narkiewicz and Śliwa, 1977)

Let $\mathcal{O}_K$ be the ring of integers of number field K. There exists a constant ρ_{typ} such that almost all non-unit elements have elasticity around ρ_{typ}, i.e., for any $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{\#\{\alpha \in \mathcal{O}_K^{\sim} \mid N(\alpha) \leq x \text{ and } \rho_{typ} - \varepsilon < \rho(\alpha) < \rho_{typ} + \varepsilon\}}{\#\{\alpha \in \mathcal{O}_K^{\sim} \mid N(\alpha) \leq x\}} = 1.$$

Similar results were obtained independently by Allen and Pleasants in 1980.

For example, in $\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$, the typical value for $\rho(\alpha)$ is $\frac{6}{5}$.

The smallest positive integer with elasticity $6/5$ in $\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$ is 160.

Indeed,

$$160 = 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 5 = \left(\frac{3}{2} - \frac{1}{2}\sqrt{-23}\right)\left(\frac{3}{2} + \frac{1}{2}\sqrt{-23}\right)\cdot 2\cdot 2\cdot 5.$$

G-solitaire

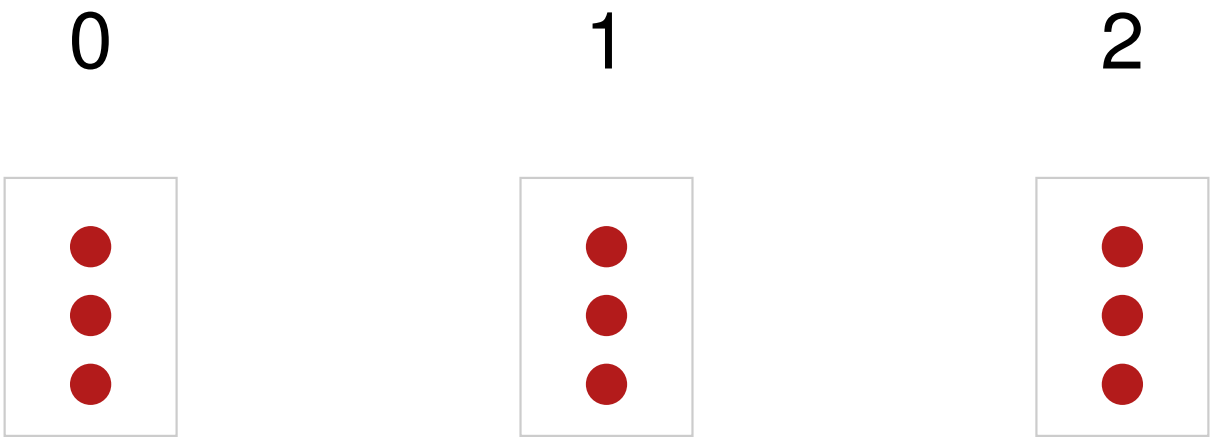
With the interest of calculating ρ_{typ}, we are going to introduce a game on additive groups G. We call this game *G-solitaire*.
Suppose G is a finite abelian group with n elements.

Rules of the Game

- We have X chips on each element of G. For simplicity, we'll assume X is an even integer.
- A move consists of grabbing chips from $g_1, g_2, \ldots, g_k \in G$ (not necessarily distinct) in such a way that $g_1 + g_2 + \cdots + g_k = 0$ and for any proper subsequence of $g_1, g_2, \ldots, g_k$, the sum is not 0.
- The goal is to clear the chips with the least number of moves. We call the minimum number of moves $\Sigma(G; X)$.

$\mathbb{Z}_3$-solitaire

- Consider the game on $\mathbb{Z}_3$.
- We have X chips on 0, X chips on 1, X chips on 2. For simplicity, assume $3 \mid X$.



Start: 3 chips on each element of $\mathbb{Z}_3$.

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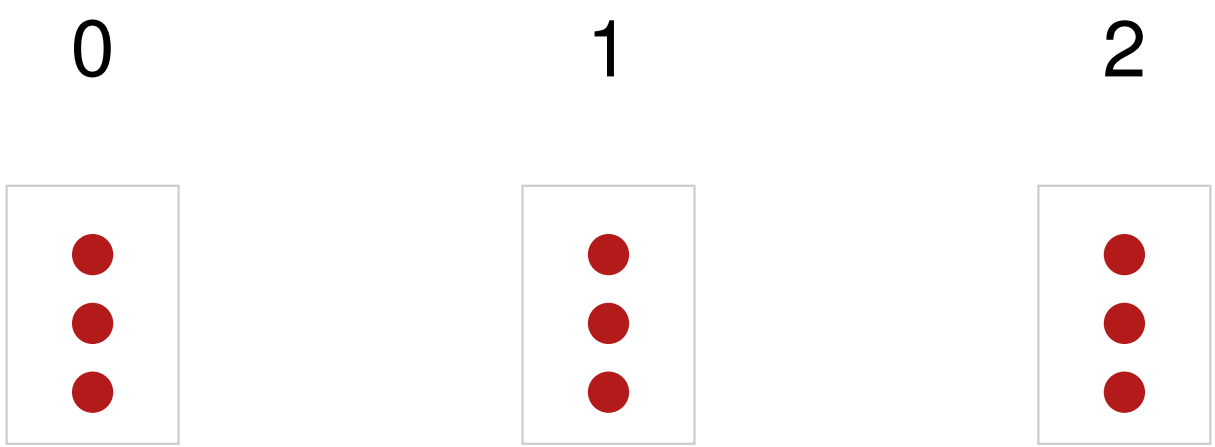

Move 1: The chips on zero can only be removed one at a time. So we will need X moves.

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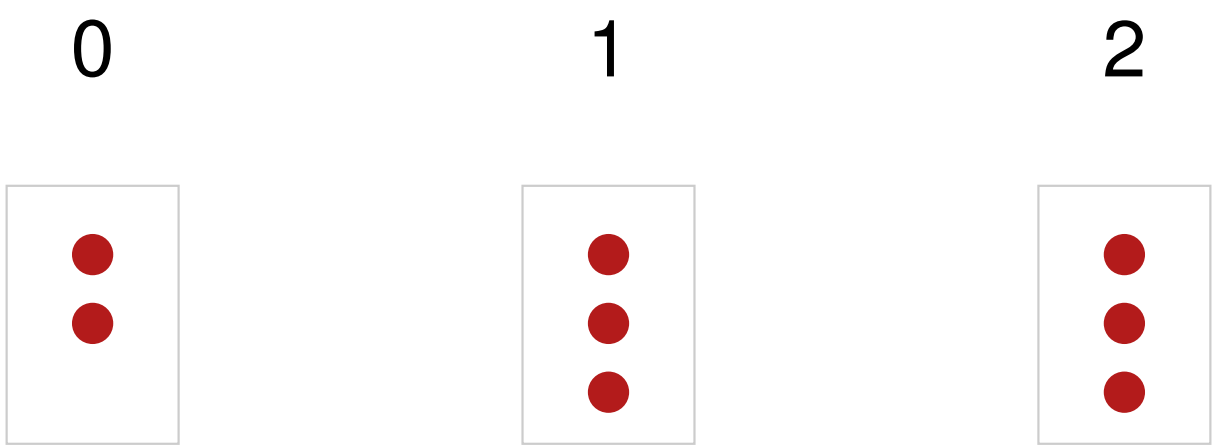

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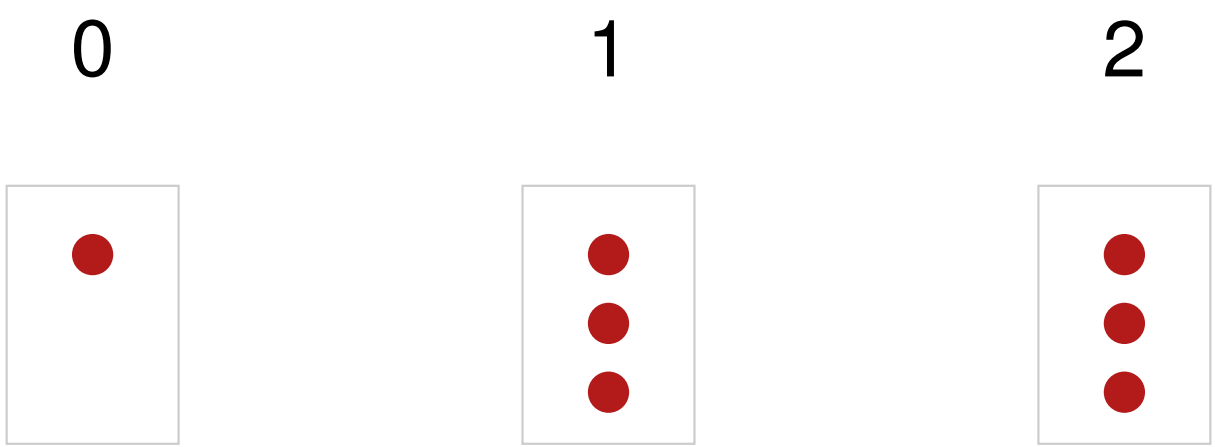

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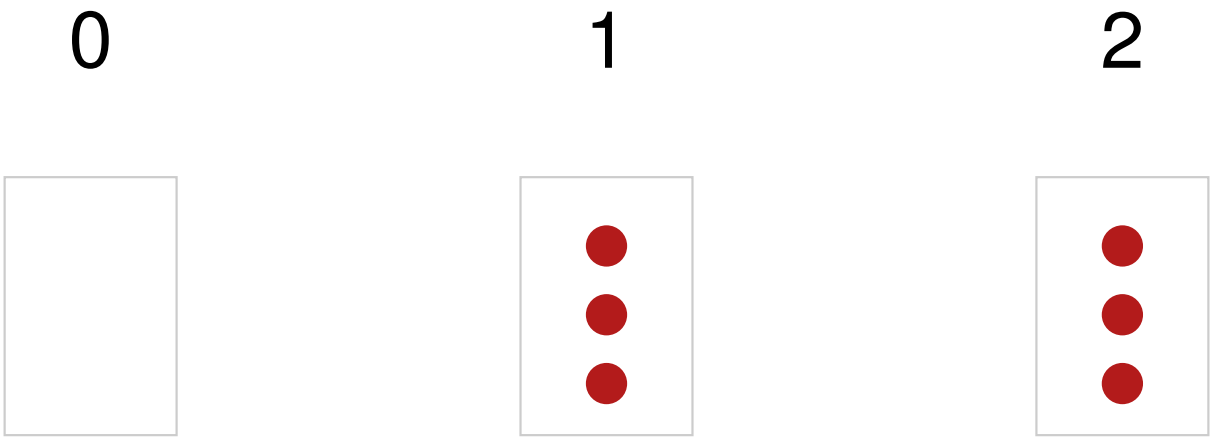

Move 2: We can clear 1s either by making a move of the form $1 + 2$ or $1 + 1 + 1$. Clearly the latter will help minimize the number of moves. We can clear in $X/3$ moves.

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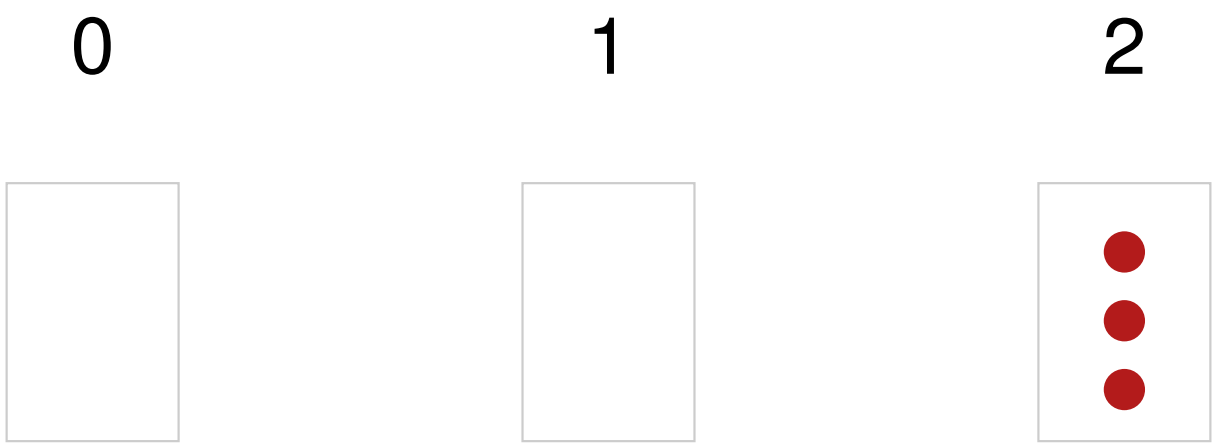

Move 3: We can clear 2s either by making a move of the form $1 + 2$ or $2 + 2 + 2$. Clearly the latter will help minimize the number of moves. We can clear in $X/3$ moves.

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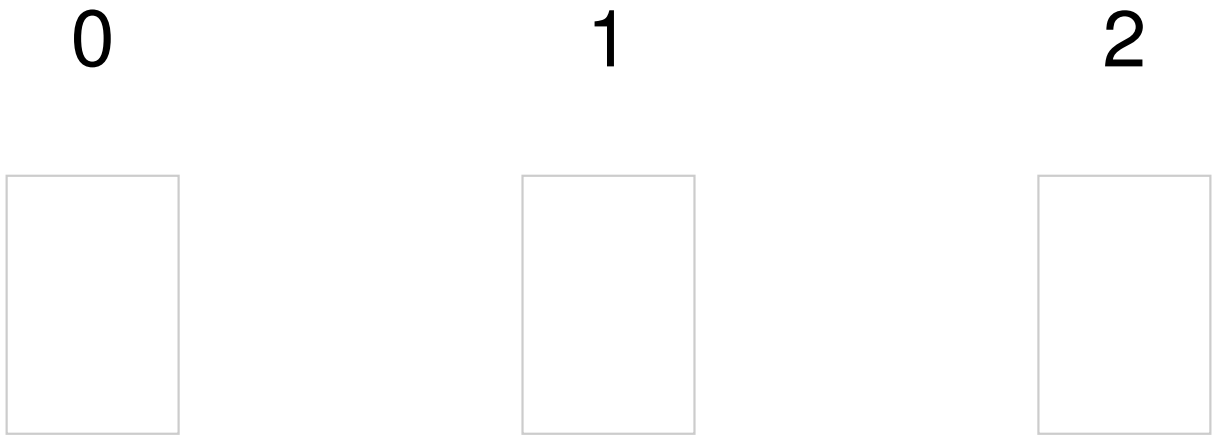

$$\Sigma(\mathbb{Z}_3; 3) = 3 + 1 + 1 = 5.$$

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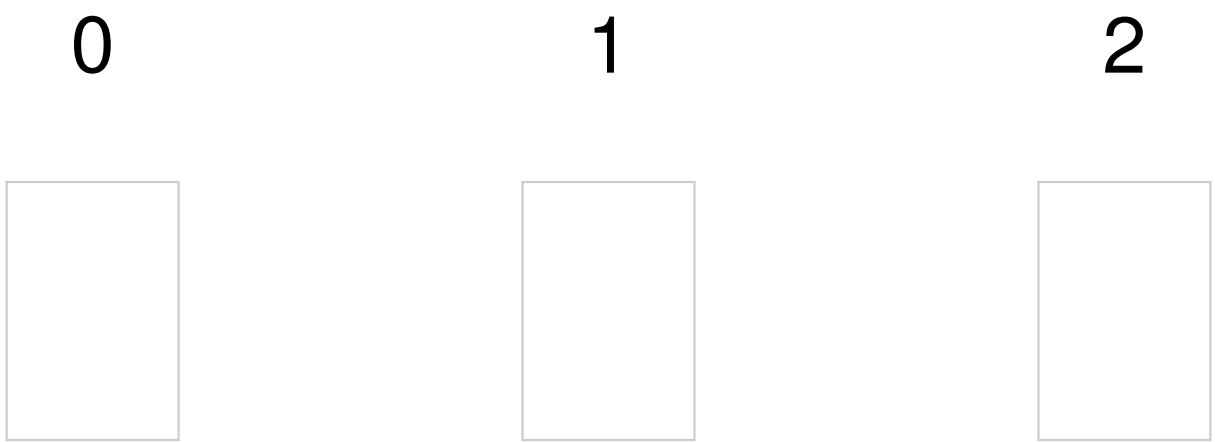

$$\Sigma(\mathbb{Z}_3; X) = X + \frac{X}{3} + \frac{X}{3} = \frac{5}{3}X.$$

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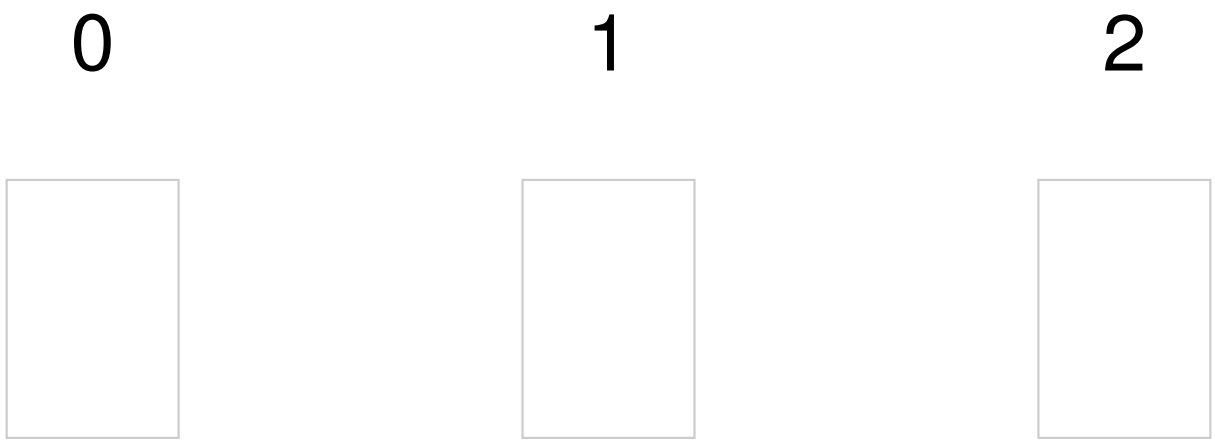
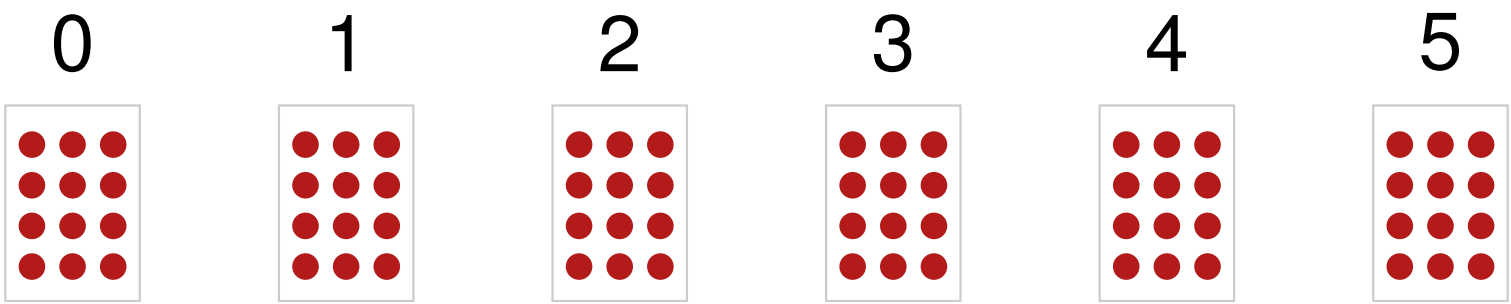


$$\boxed{\mathrm{Clr}(\mathbb{Z}_3) = \frac{5}{3}}.$$

$\mathbb{Z}_6$-solitaire

- Consider the game on $\mathbb{Z}_6$.
- We use $X = 12$ chips on each element so every move count is integral.



Start: 12 chips on each element of $\mathbb{Z}_6$.

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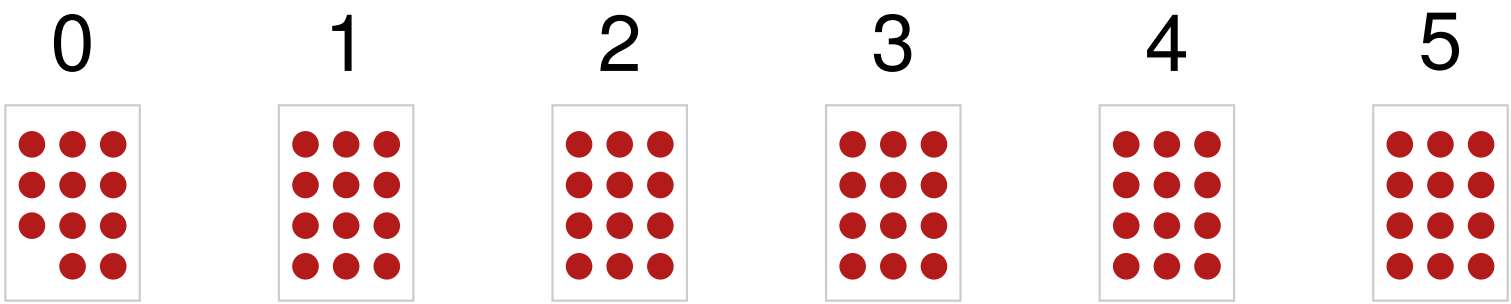

Zero move 1 of 12: Remove one chip using the singleton zero-sum move (0).

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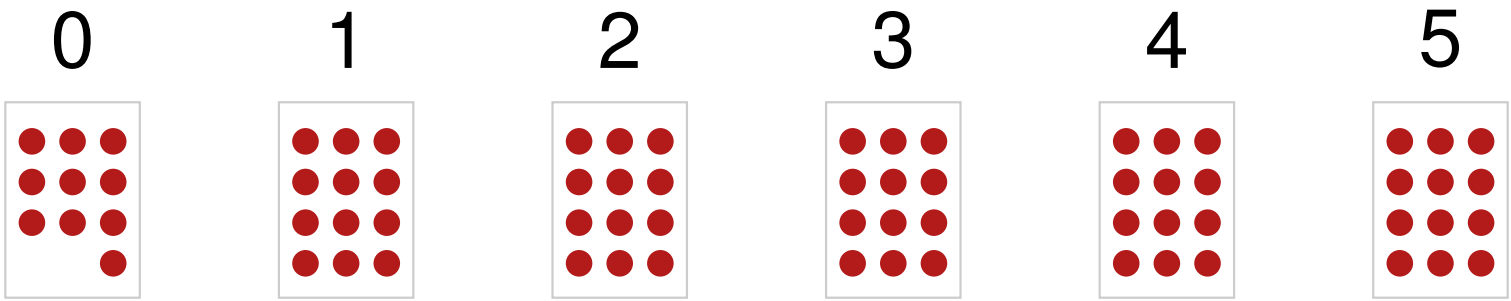

Zero move 2 of 12: Remove one chip using the singleton zero-sum move (0).

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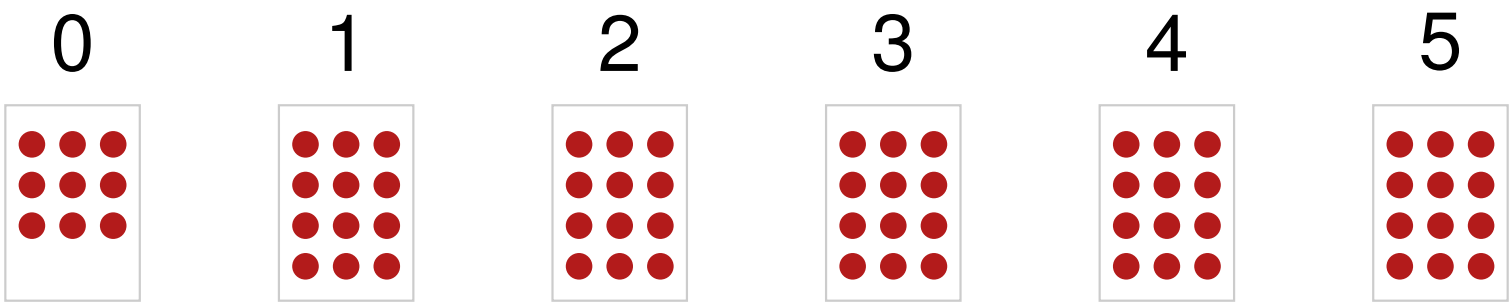

Zero move 3 of 12: Remove one chip using the singleton zero-sum move (0).

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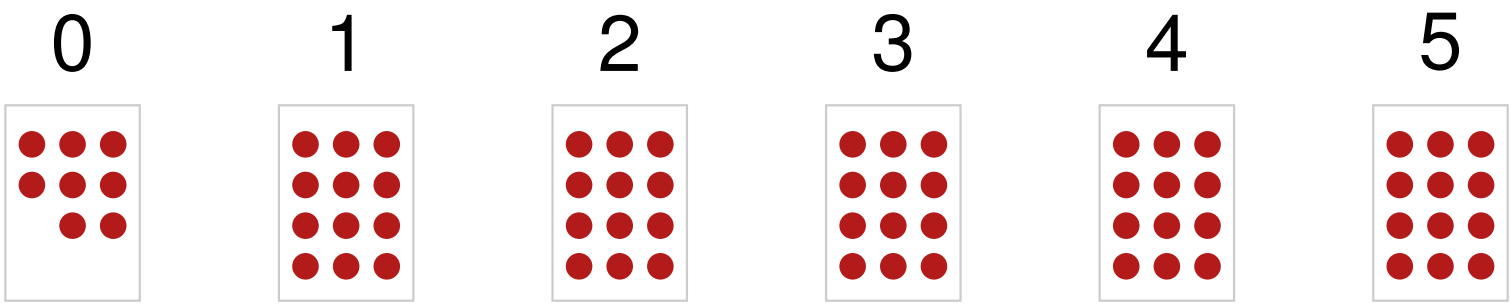

Zero move 4 of 12: Remove one chip using the singleton zero-sum move (0).

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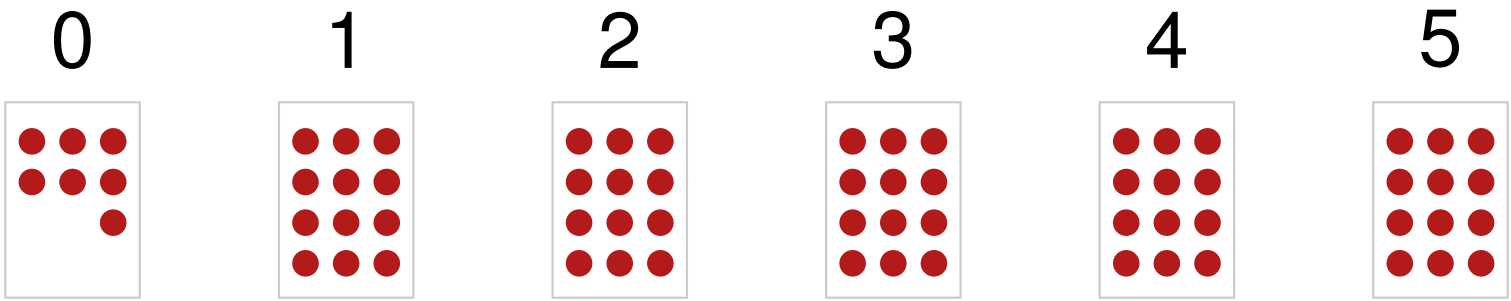

Zero move 5 of 12: Remove one chip using the singleton zero-sum move (0).

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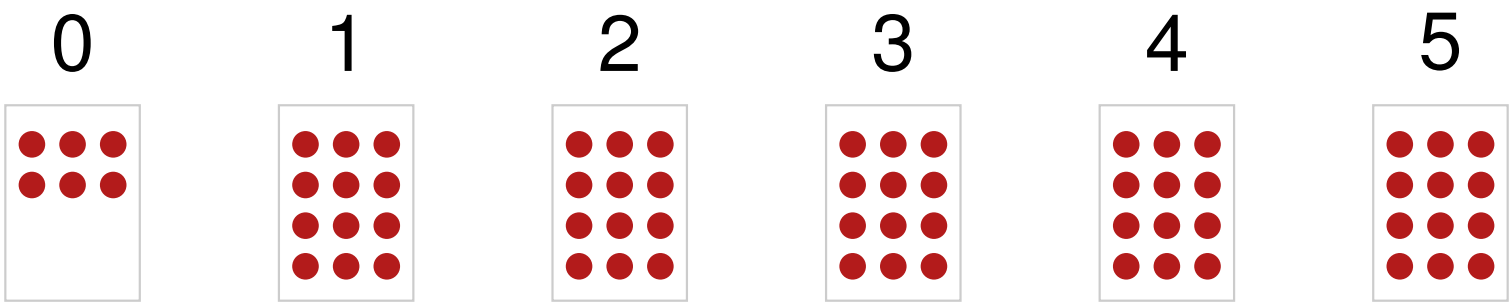

Zero move 6 of 12: Remove one chip using the singleton zero-sum move (0).

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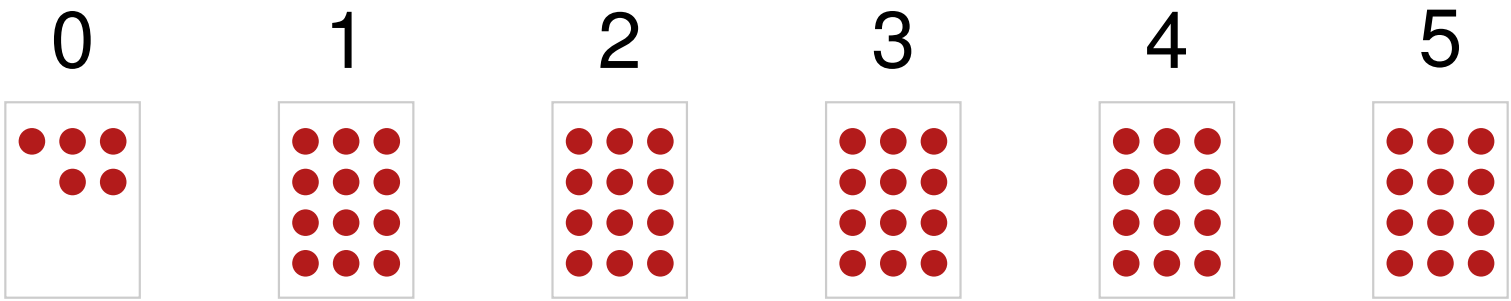

Zero move 7 of 12: Remove one chip using the singleton zero-sum move (0).

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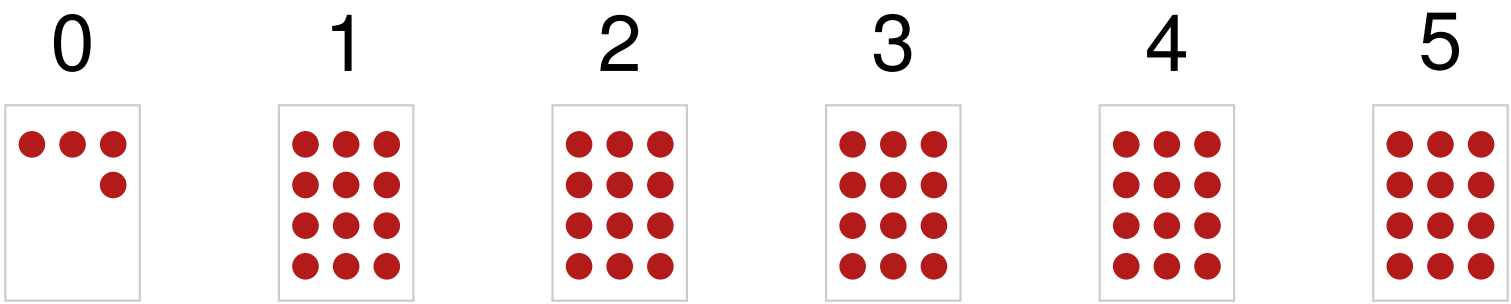

Zero move 8 of 12: Remove one chip using the singleton zero-sum move (0).

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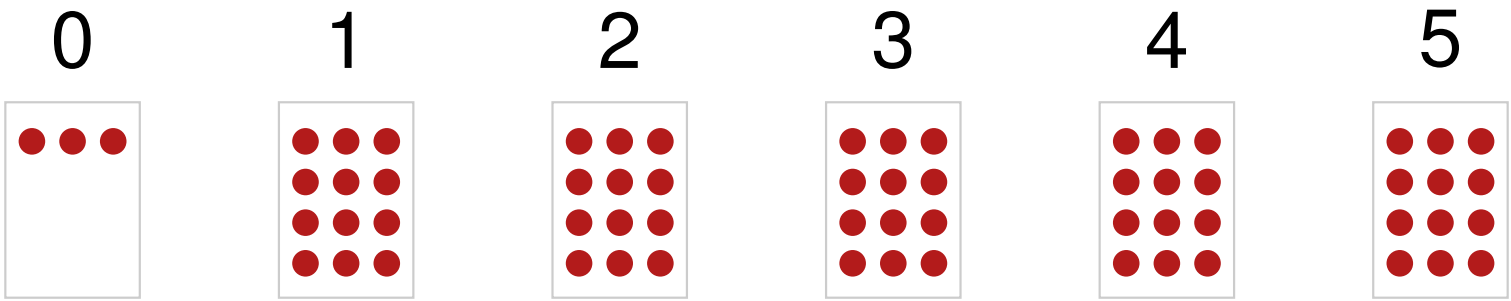

Zero move 9 of 12: Remove one chip using the singleton zero-sum move (0).

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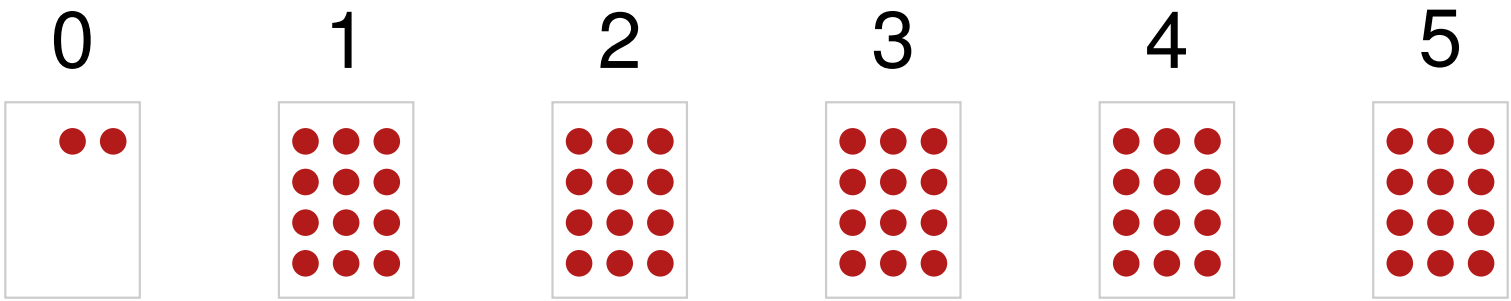

Zero move 10 of 12: Remove one chip using the singleton zero-sum move (0).

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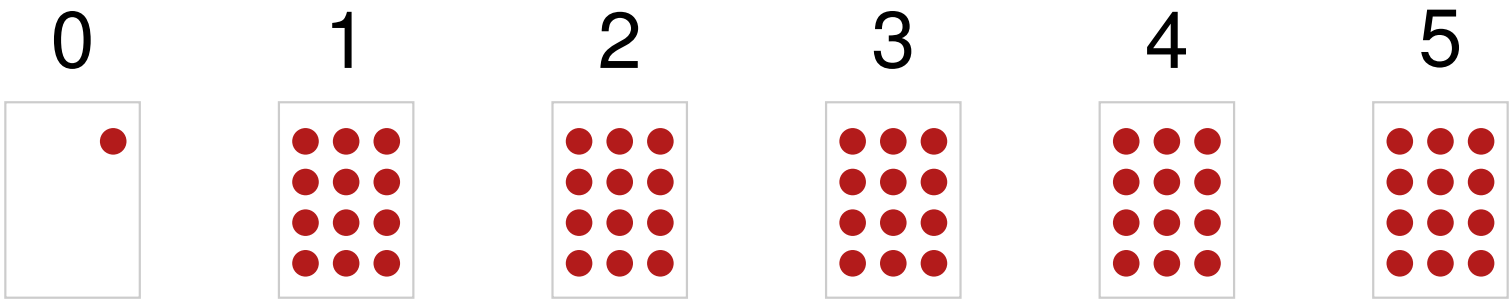

Zero move 11 of 12: Remove one chip using the singleton zero-sum move (0).

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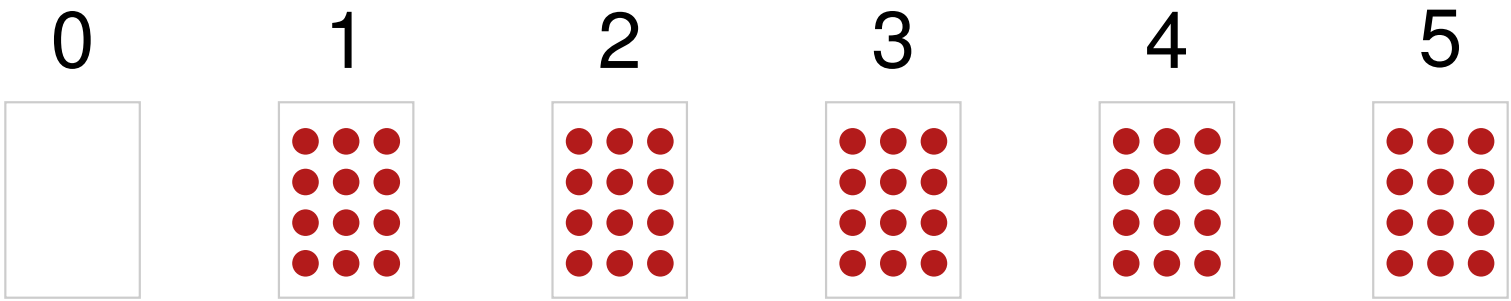

Zero move 12 of 12: Remove one chip using the singleton zero-sum move (0).

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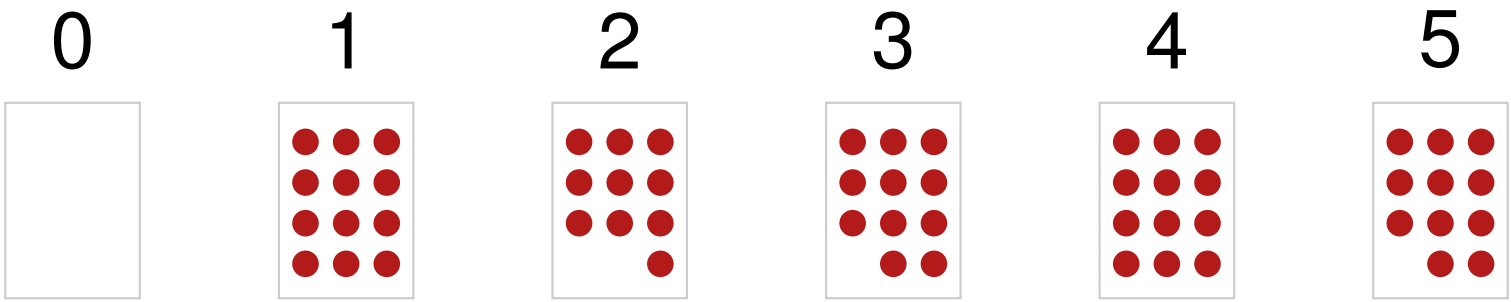

Move $2+2+3+5$, repetition 1 of 6:

$$2+2+3+5 \equiv 0 \pmod 6.$$

Remove two chips from 2, one from 3, and one from 5.

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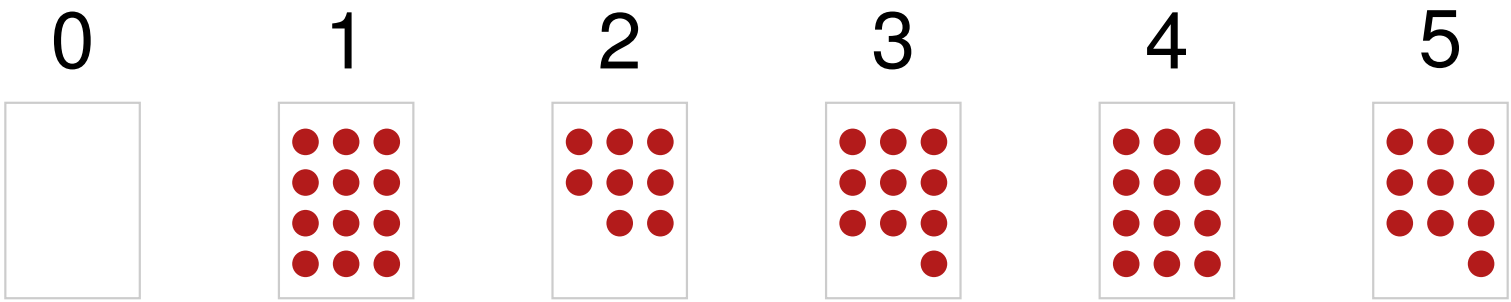


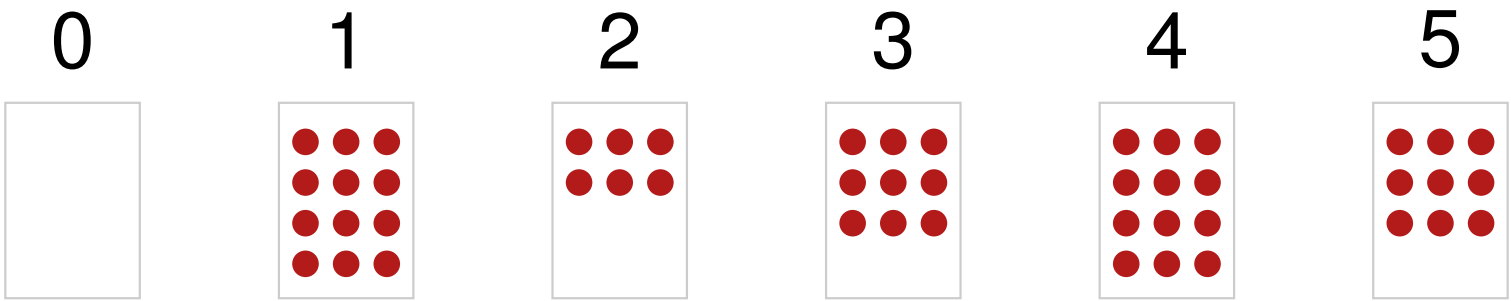
Move $2+2+3+5$, repetition 2 of 6:

$$2+2+3+5 \equiv 0 \pmod 6.$$

Remove two chips from 2, one from 3, and one from 5.

$\mathbb{Z}_6$-solitaire

- Consider the game on $\mathbb{Z}_6$.
- We use $X = 12$ chips on each element so every move count is integral.



Move $2+2+3+5$, repetition 3 of 6:

$$2+2+3+5 \equiv 0 \pmod 6.$$

Remove two chips from 2, one from 3, and one from 5.

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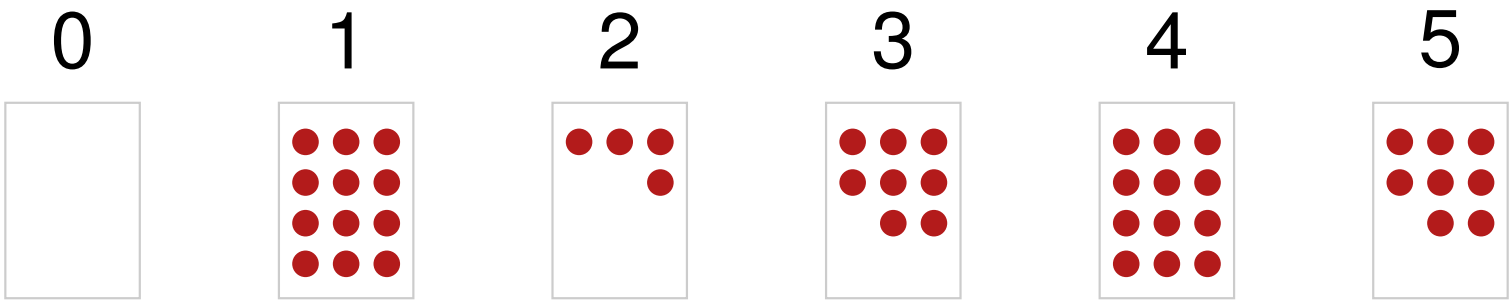

Move $2+2+3+5$, repetition 4 of 6:

$$2+2+3+5 \equiv 0 \pmod 6.$$

Remove two chips from 2, one from 3, and one from 5.

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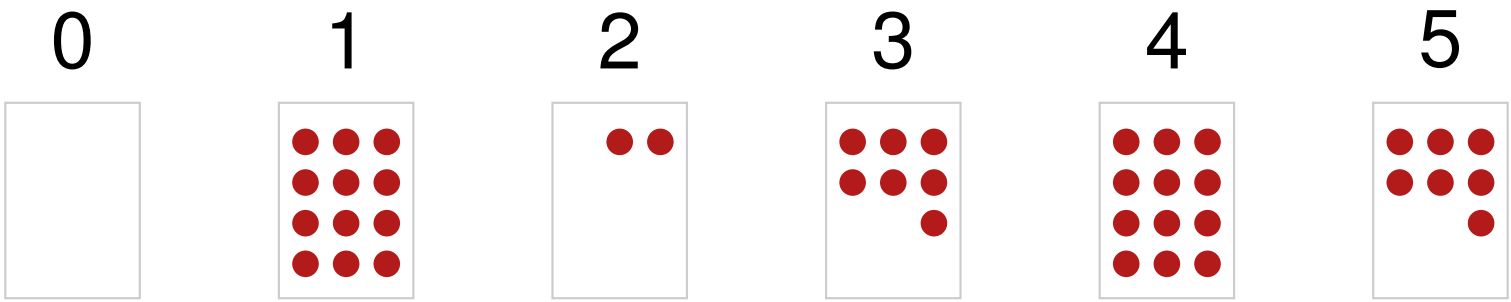

Move $2+2+3+5$, repetition 5 of 6:

$$2+2+3+5 \equiv 0 \pmod 6.$$

Remove two chips from 2, one from 3, and one from 5.

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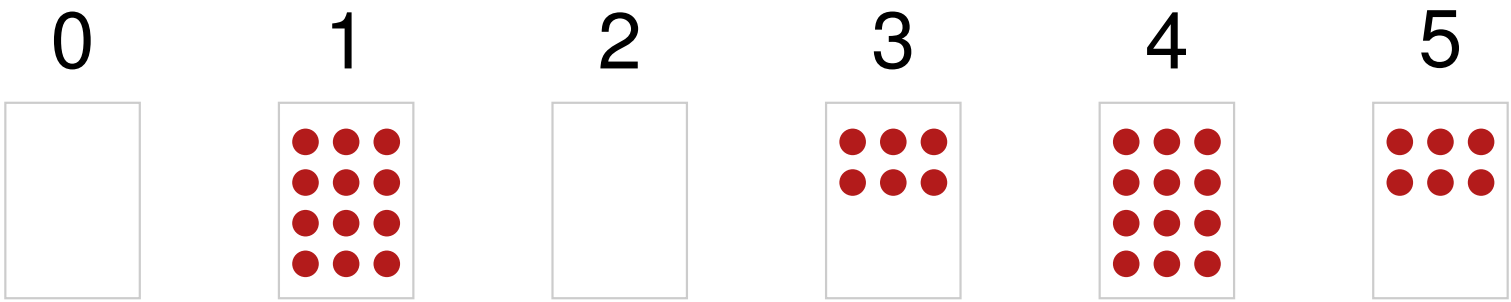

Move $2+2+3+5$, repetition 6 of 6:

$$2+2+3+5 \equiv 0 \pmod 6.$$

Remove two chips from 2, one from 3, and one from 5.

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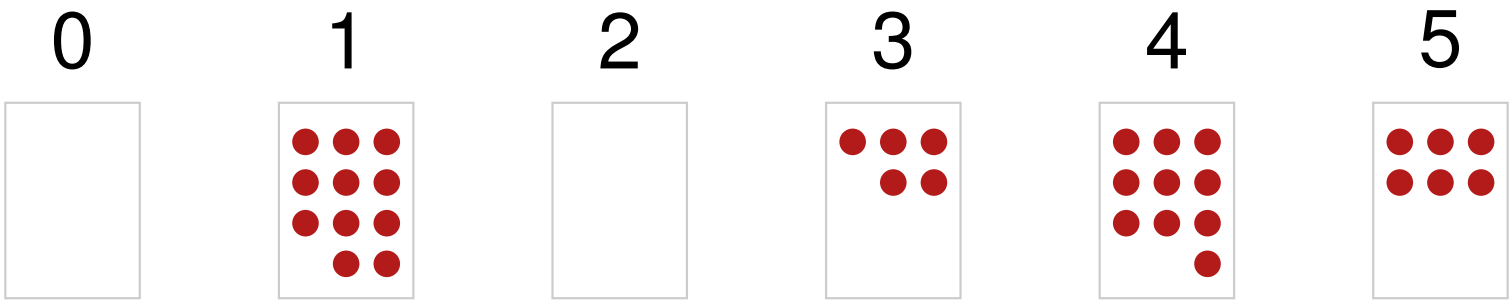

Move $4 + 4 + 3 + 1$**, repetition 1 of 6:**

$$4 + 4 + 3 + 1 \equiv 0 \pmod 6.$$

Remove two chips from 4, one from 3, and one from 1.

$\mathbb{Z}_6$-solitaire

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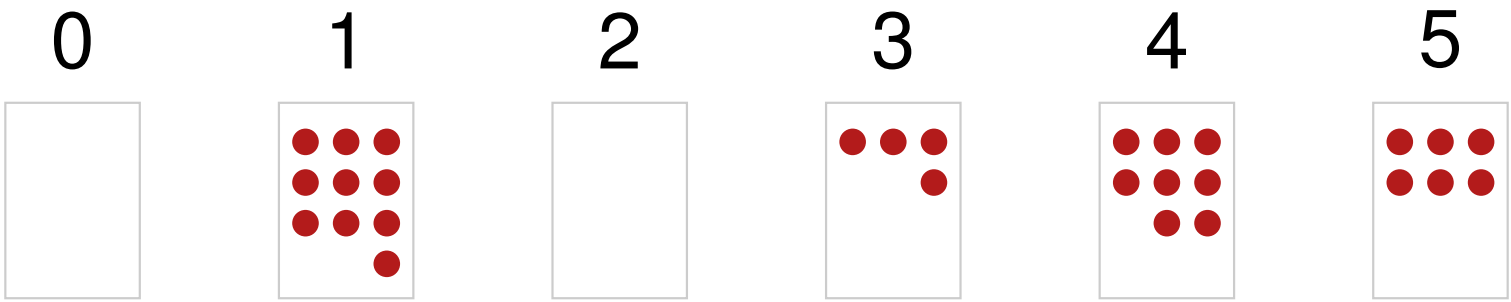

Move $4+4+3+1$, repetition 2 of 6:

$$4+4+3+1 \equiv 0 \pmod 6.$$

Remove two chips from 4, one from 3, and one from 1.

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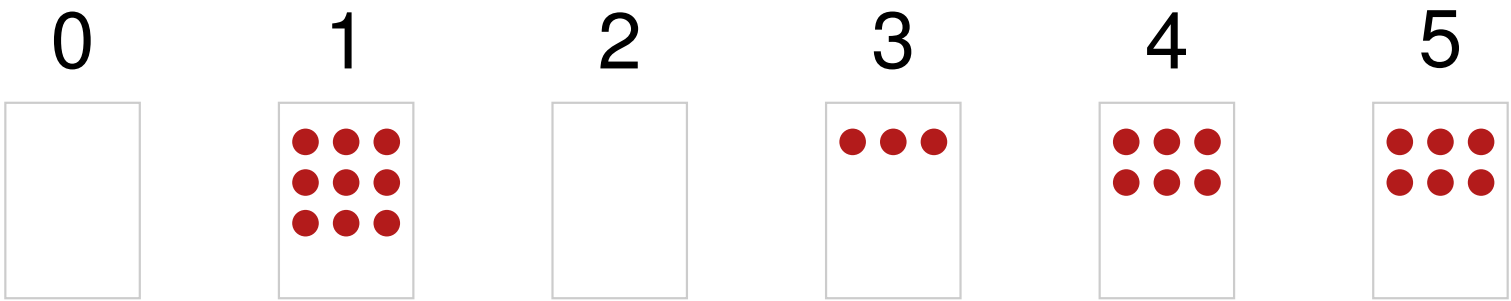

Move $4+4+3+1$, repetition 3 of 6:

$$4+4+3+1 \equiv 0 \pmod 6.$$

Remove two chips from 4, one from 3, and one from 1.

$\mathbb{Z}_6$-solitaire

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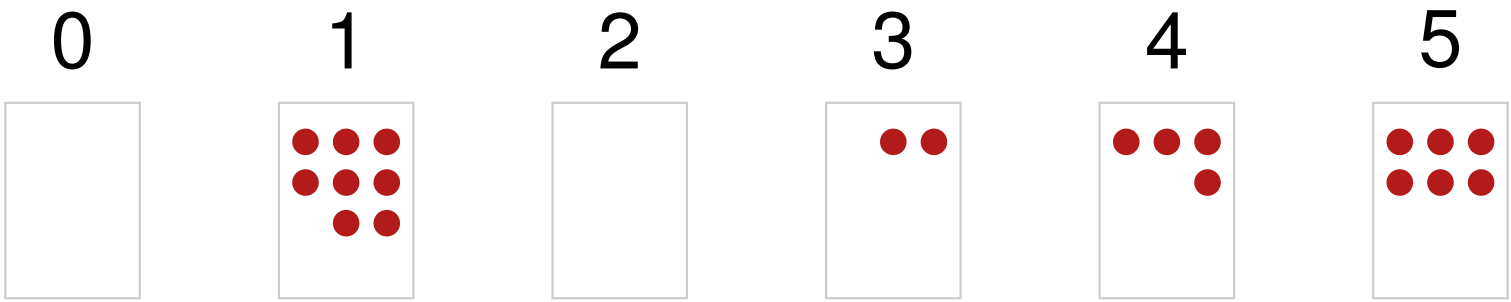

Move $4 + 4 + 3 + 1$**, repetition 4 of 6:**

$$4 + 4 + 3 + 1 \equiv 0 \pmod 6.$$

Remove two chips from 4, one from 3, and one from 1.

$\mathbb{Z}_6$-solitaire

- Consider the game on $\mathbb{Z}_6$.
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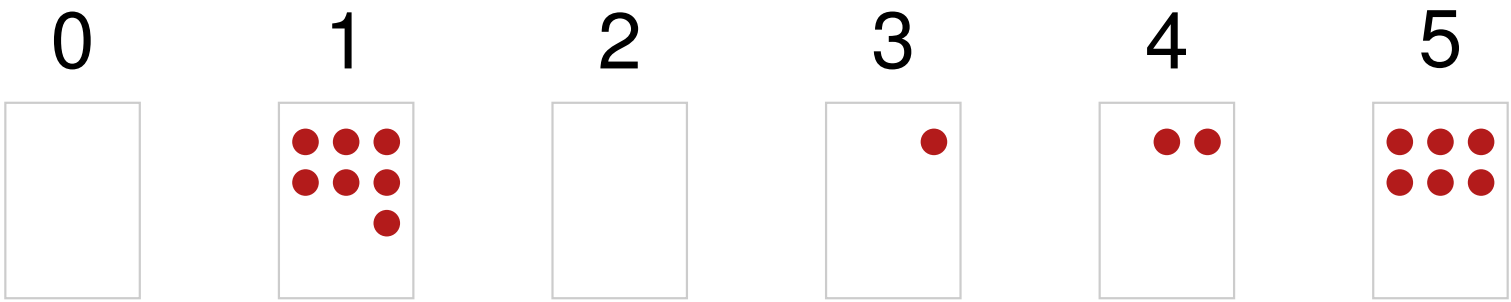

Move $4 + 4 + 3 + 1$**, repetition 5 of 6:**

$$4 + 4 + 3 + 1 \equiv 0 \pmod 6.$$

Remove two chips from 4, one from 3, and one from 1.

$\mathbb{Z}_6$-solitaire

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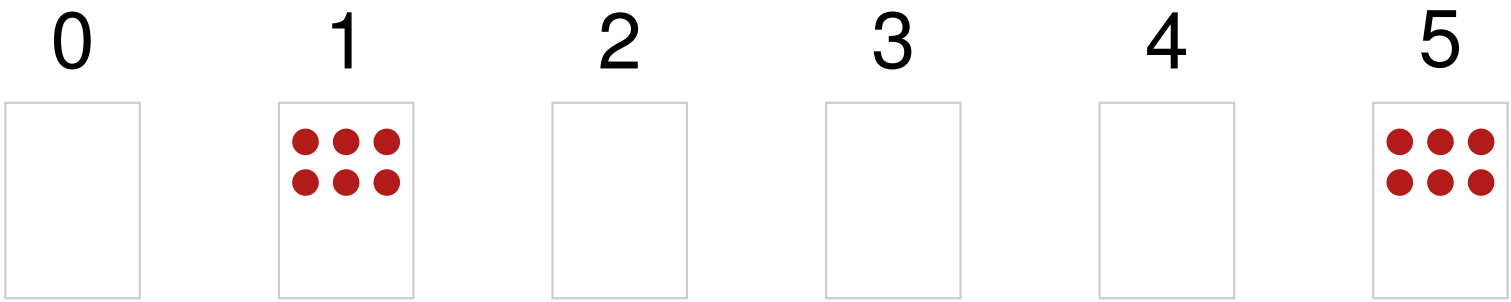

Move $4 + 4 + 3 + 1$**, repetition 6 of 6:**

$$4 + 4 + 3 + 1 \equiv 0 \pmod 6.$$

Remove two chips from 4, one from 3, and one from 1.

$\mathbb{Z}_6$-solitaire

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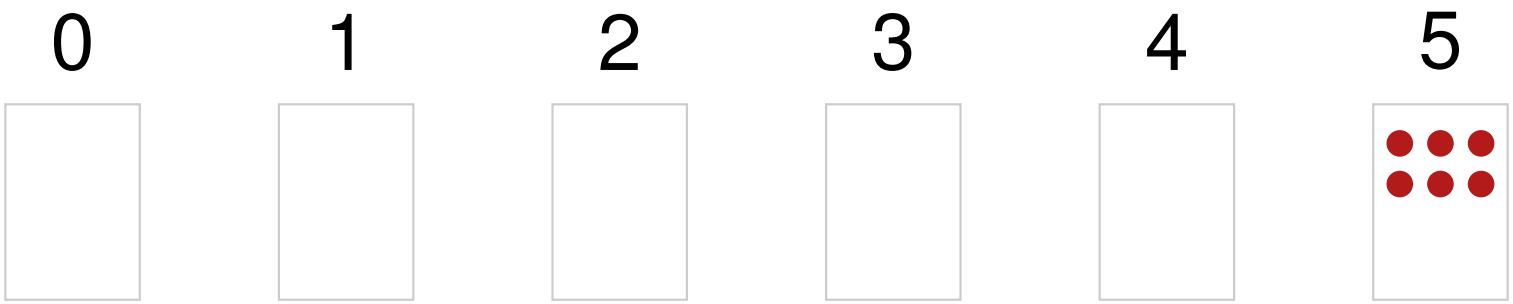

Move 25: Clear the six remaining chips at 1 in one move:

$$1 + 1 + 1 + 1 + 1 + 1 \equiv 0 \pmod 6.$$

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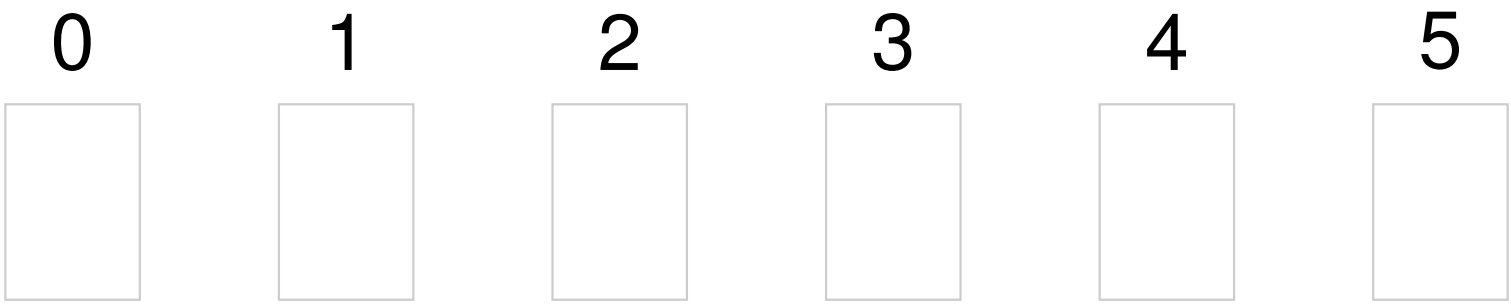

Move 26: Clear the six remaining chips at 5 in one move:

$$5 + 5 + 5 + 5 + 5 + 5 \equiv 0 \pmod 6.$$

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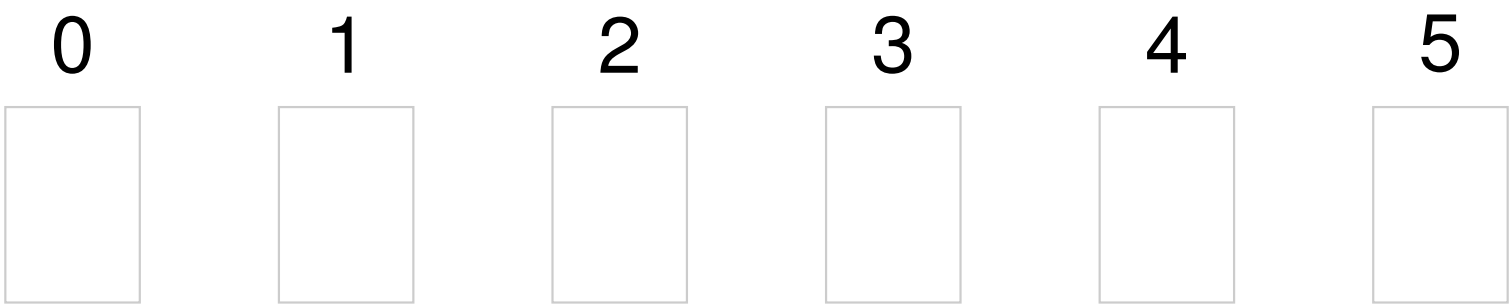

$$\Sigma(\mathbb{Z}_6; 12) = 12 + 6 + 6 + 1 + 1 = 26.$$

$\mathbb{Z}_6$-solitaire

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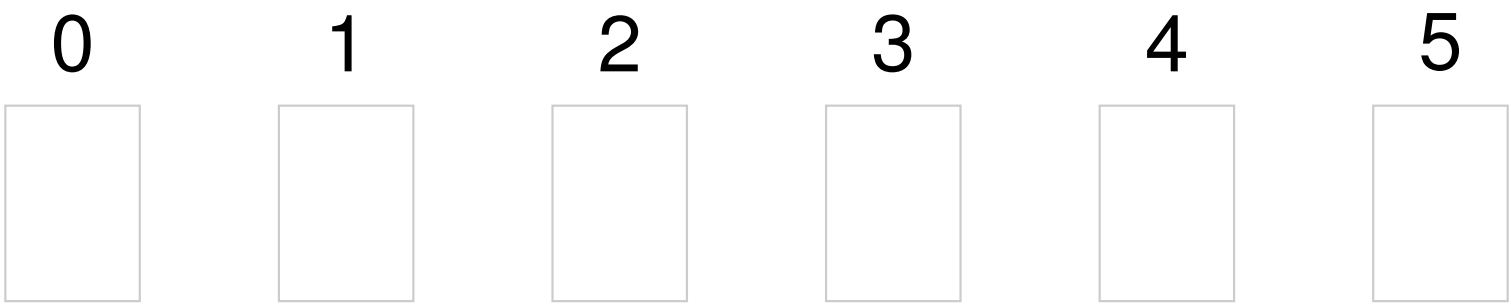

$$\Sigma(\mathbb{Z}_6; X) = X + \frac{X}{2} + \frac{X}{2} + \frac{X}{12} + \frac{X}{12} = \frac{13}{6}X.$$

$\mathbb{Z}_6$-solitaire

- Consider the game on $\mathbb{Z}_6$.
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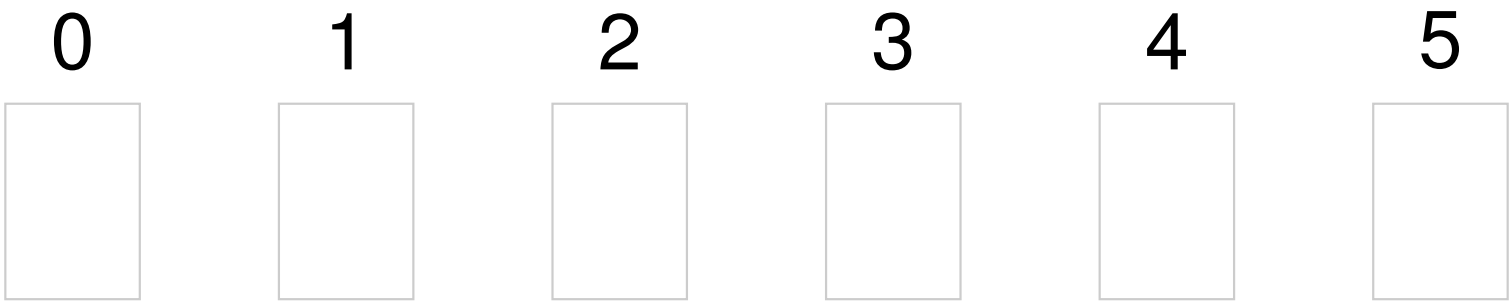

$$\boxed{\mathrm{Clr}(\mathbb{Z}_6) = \frac{13}{6}}.$$

Defining the Clearing constant

It's not hard to show that the minimum number of moves, $\Sigma(G; X)$, in G-solitaire is asymptotic to a constant multiple of X, so we can define the clearing number $\mathrm{Clr}(G)$ to be

$$\mathrm{Clr}(G) = \lim_{X \to \infty} \frac{\Sigma(G; X)}{X}.$$

We've shown

$$\mathrm{Clr}(\mathbb{Z}_3) = \frac{5}{3} \quad \text{and} \quad \mathrm{Clr}(\mathbb{Z}_6) = \frac{13}{6}.$$

- Allen and Pleasants, in 1980, found formulas for $\mathrm{Clr}\left((\mathbb{Z}_{p^r})^s\right)$, where p is prime and r, s are positive integers. They also found $\mathrm{Clr}(\mathbb{Z}_6)$ and $\mathrm{Clr}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

Connecting the clearing constant to typical elasticities

The following theorem connects the dots between our G-solitaire game and the typical elasticity of a ring of integers.

Theorem

Let K be a number field with ring of integers $\mathcal{O}_K$. Let G be the class group of $\mathcal{O}_K$. Then the typical elasticity, ρ_{typ} of $\mathcal{O}_K$ is

$$\rho_{typ} = \frac{1 + |G|}{2\mathrm{Clr}(G)}.$$

Example: Since $\mathbb{Z}_3$ is the class group of $\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$, then the typical elasticity in $\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$ is

$$\rho_{\text{typ}} = \frac{1+3}{2 \cdot \frac{5}{3}} = \frac{6}{5}.$$

Updating our Example Table

Field	Ring	Cl G	Cl $\#$	$\mathrm{Clr}(\mathrm{G})$	$\rho(\mathcal{O}_K)$	ρ_{typ}
$\mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}_1$	1	1	1	1
$\mathbb{Q}(i)$	$\mathbb{Z}[i]$	$\mathbb{Z}_1$	1	1	1	1
$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Z}[\sqrt{-5}]$	$\mathbb{Z}_2$	2	$\frac{3}{2}$	1	1
$\mathbb{Q}(\sqrt{-23})$	$\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$	$\mathbb{Z}_3$	3	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{6}{5}$
$\mathbb{Q}(\sqrt{-26})$	$\mathbb{Z}\left[\sqrt{-26}\right]$	$\mathbb{Z}_6$	6	$\frac{13}{6}$	3	$\frac{21}{13}$

Clearing constants known before our work

As mentioned before, for some small groups, the clearing constant was known, and also for groups of the form $\left(\mathbb{Z}_{p^r}\right)^s$. The known clearing constants (due to Allen and Pleasants) are:

- $$\mathrm{Clr}\left(\mathbb{Z}_2 \times \mathbb{Z}_4\right) = \frac{8}{3}.$$
- $$\mathrm{Clr}\left(\mathbb{Z}_6\right) = \frac{13}{6}.$$
- $$\mathrm{Clr}\left(\left(\mathbb{Z}_{p^r}\right)^s\right) = \begin{cases} 1 + \frac{p^{rs}(1-p^{-s})}{s(p^r-1)+1} r & s = 1 \\ 1 + \frac{p^{rs}(1-p^{-s})}{s(p^r-1)+1} \cdot \frac{1-p^{-(s-1)r}}{1-p^{-(s-1)}} & s > 1 \end{cases}.$$

Clearing Constants Due to us

Theorem

For every prime p,

$$\mathrm{Clr}\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right) = 1 + \frac{2p^2 - p - 1}{2p - 1}.$$

For example, in the case of $p = 2$ we recover the one Allen and Pleasants calculated: $\mathrm{Clr}\left(\mathbb{Z}_2 \times \mathbb{Z}_4\right) = \frac{8}{3}$.

Theorem

For every pair of distinct primes p and q,

$$\mathrm{Clr}\left(\mathbb{Z}_{pq}\right) = 3 - \frac{1}{p} - \frac{1}{q}.$$

For example, in the case of $p = 2$ and $q = 3$ we recover the one Allen and Pleasants calculated: $\mathrm{Clr}\left(\mathbb{Z}_6\right) = \frac{13}{6}$.

Funny Example

- Consider the number field $\mathbb{Q}(\sqrt{-35008251})$.
- Its ring of integers is $\mathbb{Z}\left[\frac{1+\sqrt{-35008251}}{2}\right]$.
- Its class group is $\mathbb{Z}_{2026}$.
- Its Davenport constant is 2026, so the elasticity of the ring of integers is $2026/2 = 1013$.
- Using our theorem, we can calculate that the clearing constant of $\mathbb{Z}_{2026}$ is

$$3 - \frac{1}{2} - \frac{1}{1013} = \frac{5063}{2026}.$$

- Therefore, the typical elasticity in this ring of integers is

$$\frac{1 + 2026}{2 \cdot \frac{5063}{2026}} = \frac{2053351}{5063}.$$

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Illustrating some ideas for the proof regarding $\mathbb{Z}_{pq}$

- Let's try to find the clearing constant for $\mathbb{Z}_6$.
- We already showed before that there's a construction to get an upper bound of $\frac{13}{6}$. Now we need to show that it is also a lower bound.
- Define the cross-number function κ as follows:

g	0	1	2	3	4	5
$\kappa(g)$	1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

- For a list of numbers S, let

$$\kappa(S) = \sum_{s \in S} \kappa(s).$$

- For example, the move $2+2+3+5$ would be $S = \{2,2,3,5\}$ and

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Proof continuation

- Suppose S is a sequence for a possible move in G-solitaire that contains the number 3.
  - If $S = \{3, 3\}$, then $\kappa(S) = 1$, so suppose $S \neq \{3, 3\}$.
  - Then S only contains one 3.
  - It's easy to show that the most even numbers S can contain is 2 so

$$\kappa(S) \leq \frac{1}{2} + \frac{2}{3} + \frac{1}{6} = \frac{4}{3}.$$

- Suppose S does not contain 3.
  - By parity, the number of units is even.
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  - By parity, the number of units is even.
  - Any two units can be replaced with an even number without changing the value of κ.
  - The number of even numbers is at most 3.
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Proof continuation

- Suppose S is a sequence for a possible move in G-solitaire that contains the number 3.
  - If $S = \{3, 3\}$, then $\kappa(S) = 1$, so suppose $S \neq \{3, 3\}$.
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Proof continuation

- If we consider the sequence of all chips, the total cross number is

$$X\left(1+\frac{2}{3}+\frac{1}{2}+\frac{2}{6}\right)=\frac{5}{2}X.$$

- We can make at most X moves S satisfying $\kappa(S)=\frac{4}{3}$ (because we only have X 3's), every other move is at best $\kappa(S)=1$.
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Final Slide!

Field	Ring	Cl G	Cl $\#$	$\mathrm{Clr}(\mathrm{G})$	$\rho(\mathcal{O}_K)$	ρ_{typ}
$\mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}_1$	1	1	1	1
$\mathbb{Q}(i)$	$\mathbb{Z}[i]$	$\mathbb{Z}_1$	1	1	1	1
$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Z}[\sqrt{-5}]$	$\mathbb{Z}_2$	2	$\frac{3}{2}$	1	1
$\mathbb{Q}(\sqrt{-23})$	$\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]$	$\mathbb{Z}_3$	3	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{6}{5}$
$\mathbb{Q}(\sqrt{-26})$	$\mathbb{Z}\left[\sqrt{-26}\right]$	$\mathbb{Z}_6$	6	$\frac{13}{6}$	3	$\frac{21}{13}$
$\mathbb{Q}(\sqrt{-35008251})$	$\mathbb{Z}\left[\frac{1+\sqrt{-35008251}}{2}\right]$	$\mathbb{Z}_{2026}$	2026	$\frac{5063}{2026}$	1013	$\frac{2053351}{5063}$

Thank you!