

Homework 1 Solutions

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1 Section 1.2

Problem 1. (Exercise 1)

Let $\Omega = \{a, b, c\}$ be a sample space. Let

$$m(a) = \frac{1}{2}, \quad m(b) = \frac{1}{3}, \quad m(c) = \frac{1}{6}.$$

Find the probabilities for all eight subsets of Ω .

Solution 1.

$$\begin{aligned}\mathbb{P}(\emptyset) &= 0, \\ \mathbb{P}(\{a\}) &= \frac{1}{2}, \\ \mathbb{P}(\{b\}) &= \frac{1}{3}, \\ \mathbb{P}(\{c\}) &= \frac{1}{6}, \\ \mathbb{P}(\{a, b\}) &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \\ \mathbb{P}(\{a, c\}) &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \\ \mathbb{P}(\{b, c\}) &= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}, \\ \mathbb{P}(\{a, b, c\}) &= 1.\end{aligned}$$

Problem 2. (Exercise 6)

A die is loaded in such a way that the probability of each face turning up is proportional to the number of dots on that face. (For example, a six is three times as probable as a two.) What is the probability of getting an even number in one throw?

Solution 2. We have $1 + 2 + 3 + 4 + 5 + 6 = 21$, so the probability of getting an even number is

$$\frac{2 + 4 + 6}{21} = \frac{12}{21} = \frac{4}{7}.$$

Problem 3. (Exercise 10)

For a bill to come before the president of the United States, it must be passed by both the House of Representatives and the Senate. Assume that, of the bills presented to these two bodies, 60 percent pass the House, 80 percent pass the Senate, and 90 percent pass at least one of the two. Calculate the probability that the next bill presented to the two groups will come before the president.

Solution 3. Let H and S represent House and Senate, so $\mathbb{P}(H) = 0.6$, $\mathbb{P}(S) = 0.8$, and $\mathbb{P}(H \cup S) = 0.9$. The probability that a bill will come to the president is $\mathbb{P}(H \cap S)$. We know

$$\mathbb{P}(H \cup S) = \mathbb{P}(H) + \mathbb{P}(S) - \mathbb{P}(H \cap S),$$

so

$$\mathbb{P}(H \cap S) = \mathbb{P}(H) + \mathbb{P}(S) - \mathbb{P}(H \cup S) = 0.6 + 0.8 - 0.9 = 0.5.$$

Therefore, the probability is $1/2$.

Problem 4. (Exercise 31)

A reader of Marilyn vos Savant's column wrote in with the following question.

My dad heard this story on the radio. At Duke University, two students had received A's in chemistry all semester. But on the night before the final exam, they were partying in another state and didn't get back to Duke until it was over. Their excuse to the professor was that they had a flat tire, and they asked if they could take a make-up test. The professor agreed, wrote out a test and sent the two to separate rooms to take it. The first question (on one side of the paper) was worth 5 points, and they answered it easily. Then they flipped the paper over and found the second question, worth 95 points: "Which tire was it?" What was the probability that both students would say the same thing? My dad and I think it's 1 in 16. Is that right?

- (a) Is the answer $1/16$?
- (b) The following question was asked of a class of students.

I was driving to school today, and one of my tires went flat. Which tire do you think it was?

The responses were as follows: right front, 58%; left front, 11%; right rear, 18%; left rear, 13%. Suppose that this distribution holds in the general population, and assume that the two test-takers are randomly chosen from the general population. What is the probability that they will give the same answer to the second question?

Solution 4.

- (a) The answer is not $1/16$, it is actually $1/4$. That's because there are 4 ways for both of them to name the same tire out of the 16 possibilities.
- (b) The answer is

$$0.58^2 + 0.11^2 + 0.18^2 + 0.13^2 = 0.3978.$$

2 Section 2.2

Problem 5. (Exercise 1)

Suppose you choose at random a real number X from the interval $[2, 10]$.

- (a) Find the density function $f(x)$ and the probability of an event E for this experiment, where E is a subinterval $[a, b]$ of $[2, 10]$.
- (b) From (a), find the probability that $X > 5$, that $5 < X < 7$, and that $X^2 - 12X + 35 > 0$.

Solution 5.

- (a) The density function is

$$f(x) = \begin{cases} \frac{1}{8} & \text{if } 2 \leq x \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

The probability of an event E where $[a, b] \subseteq [2, 10]$ is

$$\int_a^b \frac{1}{8} dx = \frac{b-a}{8}.$$

(b)

$$\mathbb{P}(X > 5) = \int_5^{10} \frac{1}{8} dx = \frac{5}{8}.$$

$$\mathbb{P}(5 < X < 7) = \int_5^7 \frac{1}{8} dx = \frac{2}{8} = \frac{1}{4}.$$

Let's solve $X^2 - 12X + 35 > 0$. We have $X^2 - 12X + 36 = (X - 6)^2$, so $X^2 - 12X + 35 > 0$ is equivalent to $(X - 6)^2 > 1$, so $X < 5$ or $X > 7$. Therefore,

$$\mathbb{P}(X^2 - 12X + 35 > 0) = 1 - \mathbb{P}(5 \leq X \leq 7) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Problem 6. (Exercise 4)

Suppose you throw a dart at a circular target of radius 10 inches. Assuming that you hit the target and that the coordinates of the outcomes are chosen at random, find the probability that the dart falls

- (a) within 2 inches of the center.
- (b) within 2 inches of the rim.
- (c) within the first quadrant of the target.
- (d) within the first quadrant and within 2 inches of the rim.

Solution 6.

- (a) The probability is

$$\frac{\pi(2^2)}{\pi(10)^2} = \frac{4}{100}.$$

- (b) The probability is

$$\frac{\pi(10)^2 - \pi(8)^2}{\pi(10)^2} = \frac{36}{100}.$$

- (c) The probability is

$$\frac{1}{4}.$$

- (d) The probability is

$$\frac{1}{4} \cdot \frac{36}{100} = \frac{9}{100}.$$

Problem 7. (Exercise 6)

Assume that a new light bulb will burn out after t hours, where t is chosen from $[0, \infty)$ with an exponential density

$$f(t) = \lambda e^{-\lambda t}.$$

In this context, λ is often called the failure rate of the bulb.

- (a) Assume that $\lambda = 0.01$, and find the probability that the bulb will not burn out before T hours. This probability is often called the reliability of the bulb.
- (b) For what T is the reliability of the bulb equal to $1/2$?

Solution 7.

- (a) Let's find the probability for a general λ . What we care about is $\mathbb{P}(X > T)$, so we need to evaluate

$$\int_T^\infty \lambda e^{-\lambda t} dt.$$

Using $u = -\lambda t$, we get $du = -\lambda dt$, so

$$\int_T^\infty \lambda e^{-\lambda t} dt = - \int_{-\lambda T}^{-\infty} e^u du = \int_{-\infty}^{-\lambda T} e^u du = e^u \Big|_{-\infty}^{-\lambda T} = e^{-\lambda T}.$$

In the case that $\lambda = 0.01$ we get

$$e^{-0.01T}.$$

- (b) We need to solve

$$e^{-0.01T} = \frac{1}{2}.$$

Therefore

$$\begin{aligned} e^{-0.01T} &= \frac{1}{2} \\ -0.01T &= \log\left(\frac{1}{2}\right) \\ -0.01T &= -\log 2 \\ T &= 100 \log 2 = 69.31. \end{aligned}$$

Problem 8. (Exercise 8)

Choose independently two numbers B and C at random from the interval $[0, 1]$ with uniform density. Note that the point (B, C) is then chosen at random in the unit square. Find the probability that

- (a) $B + C < \frac{1}{2}$.
- (b) $BC < \frac{1}{2}$.
- (d) $\max\{B, C\} < \frac{1}{2}$.
- (h) $B^2 + C^2 \leq \frac{1}{2}$.

Solution 8.

- (a) We need $B + C < \frac{1}{2}$, so it creates a triangle of area $1/8$ as seen in Figure 1.

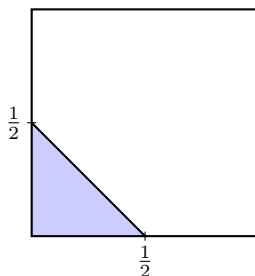


Figure 1: The event $B + C < \frac{1}{2}$.

Therefore, the probability is

$$\frac{1}{8}.$$

- (b) In this problem, the key is the equation $y = \frac{1}{2x}$. It intersects the unit square at $(1/2, 1)$ and at $(1, 1/2)$. The shaded area is easier to calculate by looking at the area not shaded and doing 1 minus that. Therefore, the probability is

$$1 - \int_{\frac{1}{2}}^1 \left(1 - \frac{1}{2x}\right) dx = 1 - \left(x - \frac{1}{2} \log x\right) \Big|_{\frac{1}{2}}^1 = \frac{1}{2} + \frac{1}{2} \log 2.$$

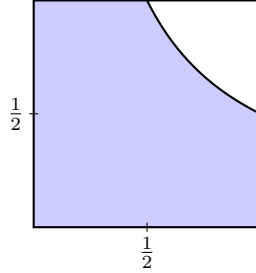


Figure 2: The event $BC < \frac{1}{2}$.

- (d) This one is equivalent to requiring $B < 1/2$ and $C < 1/2$, so then we have that the probability is $\frac{1}{4}$.

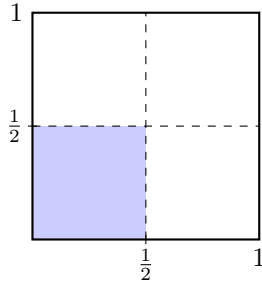


Figure 3: The event $\max\{B, C\} < \frac{1}{2}$.

- (h) The condition $B^2 + C^2 \leq \frac{1}{2}$ describes a quarter circle of radius $\frac{1}{\sqrt{2}}$, so it's area is

$$\frac{\pi}{8}.$$

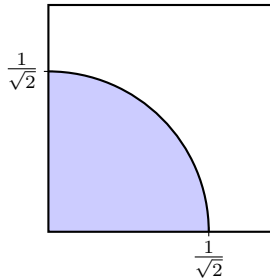


Figure 4: The event $B^2 + C^2 \leq \frac{1}{2}$.