

Homework 3 Solutions

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1 Section 4.1

Problem 1. (Exercise 1)

Assume that E and F are two events with positive probabilities. Show that if $P(E|F) = P(E)$, then $P(F|E) = P(F)$.

Solution 1. Suppose $P(E|F) = P(E)$. We know (using that $P(F) > 0$) that

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

Since $P(E) = P(E|F)$, then

$$P(E)P(F) = P(E \cap F).$$

But then, using that $P(E) \neq 0$, we have

$$P(F) = \frac{P(E \cap F)}{P(E)} = P(F|E).$$

Problem 2. (Exercise 2)

A coin is tossed three times. What is the probability that exactly two heads occur, given that

- (a) the first outcome was a head?
- (b) the first outcome was a tail?
- (c) the first two outcomes were heads?
- (d) the first two outcomes were tails?
- (e) the first outcome was a head and the third outcome was a head?

Solution 2.

- (b) This is equivalent to what is the probability that the last two coin tosses are heads. Therefore, it is $1/4$.
- (c) This is equivalent to the last coin toss is tails, which has probability is $1/2$.
- (d) This is impossible, so the probability is 0.
- (e) This is equivalent to the second outcome being tails, so the probability is $1/2$.

Problem 3. (Exercise 4)

A card is drawn at random from a deck of cards. What is the probability that

- (a) it is a heart, given that it is red?
- (b) it is higher than a 10, given that it is a heart? (Interpret J, Q, K, A as 11, 12, 13, 14.)

(c) it is a jack, given that it is red?

Solution 3.

(a) There are two red suits, hearts and diamonds. Therefore, the probability is $1/2$.

(b) There are four out of 13 values above ten, so the probability is $4/13$.

(c) The probability is $1/13$.

Problem 4. (Exercise 6)

From a deck of five cards numbered 2, 4, 6, 8, and 10, respectively, a card is drawn at random and replaced. This is done three times. What is the probability that the card numbered 2 was drawn exactly two times, given that the sum of the numbers on the three draws is 12?

Solution 4. To have 2 drawn twice and the sum equal to 12, the three draws must have been 2,2,8 in one of the three possible orders (228, 282, 822). Now we need to find all ways in which the three draws could have added to 12. We can count it manually

- $2 + 2 + 8$ happens in 3 ways ($2 + 2 + 8, 2 + 8 + 2, 8 + 2 + 2$).
- $2 + 4 + 6$ happens in 6 ways.
- $4 + 4 + 4$ happens one way.

Therefore, there are 10 ways to get a sum of 12 and 3 of them have two 2s. The conclusion is that the probability is $\frac{3}{10}$.

Problem 5. (Exercise 12)

In Example 4.17, how large must the prior probability of cancer be to give a posterior probability of .5 for cancer given a positive test?

Solution 5. Let the prior probability be p . Using the notation from the example, we still have $P(+|\text{cancer}) = 0.99$ and $P(+|\text{not cancer}) = 0.05$. We want $P(\text{cancer}|+)$. Using Bayes theorem, we have

$$\begin{aligned} P(\text{cancer}|+) &= \frac{P(\text{cancer})P(+|\text{cancer})}{P(\text{cancer})P(+|\text{cancer}) + P(\text{not cancer})P(+|\text{not cancer})} \\ &= \frac{0.99p}{0.99p + 0.05(1-p)} = \frac{0.99p}{0.05 + 0.94p}. \end{aligned}$$

Since we want the posterior probability to be 0.5, we must solve

$$\frac{0.99p}{0.05 + 0.94p} = \frac{1}{2}.$$

Therefore

$$\begin{aligned} 1.98p &= 0.05 + 0.94p \\ 1.04p &= 0.05 \\ p &= \frac{0.05}{1.04} = \frac{5}{104} \approx 0.048. \end{aligned}$$

Problem 6. (Exercise 14)

If $P(\tilde{B}) = 1/4$ and $P(A|B) = 1/2$, what is $P(A \cap B)$?

Solution 6. We know

$$\frac{1}{2} = P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{1 - P(\tilde{B})} = \frac{A \cap B}{\frac{3}{4}}.$$

It follows that

$$P(A \cap B) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}.$$

Problem 7. (Exercise 19)

In a poker hand, John has a very strong hand and bets 5 dollars. The probability that Mary has a better hand is .04. If Mary had a better hand she would raise with probability .9, but with a poorer hand she would only raise with probability .1. If Mary raises, what is the probability that she has a better hand than John does?

Solution 7. Let A be the event that Mary has a better hand and B be the event that Mary raises. We know that $P(A) = 0.04$ and that $P(B|A) = 0.9$. What we want to calculate is $P(A|B)$. From Bayes Theorem we get

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\tilde{A})P(B|\tilde{A})} = \frac{0.04 \cdot 0.9}{0.04 \cdot 0.9 + 0.96 \cdot 0.1} = \frac{0.036}{0.132} = \frac{3}{11} = 0.2727 \dots$$

Problem 8. (Exercise 22)

One coin in a collection of 65 has two heads. The rest are fair. If a coin, chosen at random from the lot and then tossed, turns up heads 6 times in a row, what is the probability that it is the two-headed coin?

Solution 8. Let A be the event that the coin is the two-headed coin. We have $P(A) = \frac{1}{65}$ and $P(\tilde{A}) = \frac{64}{65}$. Let B be the event that the coin came up heads 6 times in a row. We know

$$P(B|A) = 1,$$

while

$$P(B|\tilde{A}) = \frac{1}{2^6} = \frac{1}{64}.$$

We are interested in the question $P(A|B)$. From Bayes theorem, we get

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\tilde{A})P(B|\tilde{A})} = \frac{\frac{1}{65} \cdot 1}{\frac{1}{65} \cdot 1 + \frac{64}{65} \cdot \frac{1}{64}} = \frac{1}{2}.$$

Problem 9. (Exercise 25)

(Johnsonbough) A coin with probability p for heads is tossed n times. Let E be the event “a head is obtained on the first toss” and F_k the event “exactly k heads are obtained.” For which pairs (n, k) are E and F_k independent?

Solution 9. We have

$$P(E) = p,$$

while for $0 \leq k \leq n$,

$$P(F_k) = \binom{n}{k} p^k q^{n-k},$$

where $q = 1 - p$. Note that we should restrict to $0 \leq k \leq n$ because if $k > n$ or $k < 0$, then $P(F_k) = 0$. (Given the question, all the pairs with $k < 0$ and $k > n$ satisfy the criterion as $P(F_k) = 0$ and the events are forced to be independent).

If $p = 0$ then all pairs of (n, k) satisfy the criterion because $P(E) = 0$. If $p = 1$, then if $n \neq k$ we have $P(F_k) = 0$ because q^{n-k} would be 0 unless $n = k$. In the case $p = 1$ and $n = k$, we have that both $P(E) = P(F_k) = 1$. Therefore, when $p = 0$ or $p = 1$, all pairs (n, k) work.

Let's now restrict to the interesting cases, which are $0 < p < 1$ and $0 \leq k \leq n$. Fix p . We now have $P(E) > 0$ and $P(F_k) > 0$, so we can reduce the problem to solving the equation

$$P(F_k|E) = P(F_k).$$

Let's first calculate $P(F_k|E)$. If we know the first coin toss is heads, then when $k = 0$, $P(F_k|E) = 0 \neq P(F_k)$, which means F_k and E are not independent. Let's now consider $k \geq 1$. If the first coin toss is heads, we must get exactly $k - 1$ heads in the other $n - 1$ tosses, so we get $P(F_{k-1})$. Since p, q are not zero, we can divide both sides by $p^{k-1} q^{n-k}$ and get For independence, we need to solve

$$\binom{n-1}{k-1} p^{k-1} q^{n-k} = \binom{n}{k} p^k q^{n-k}.$$

Since p, q are not zero, we can divide both sides by $p^{k-1}q^{n-k}$ and get

$$\binom{n-1}{k-1} = \binom{n}{k} p.$$

But then

$$\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{k!(n-k)!} p$$

$$k = np.$$

So, when n is fixed, the only pair of (n, k) for which E and F_k are independent is the one where $k = np$ (when np is an integer). For example, for $p = 0.3$, the pairs that work are those of the form $(10k, 3k)$, so $(10, 3), (20, 6), (30, 9), \dots$

Problem 10. (Exercise 39)

Assume that the random variables X and Y have the joint distribution given in Table 4.6.

	$Y = -1$	$Y = 0$	$Y = 1$	$Y = 2$
$X = -1$	0	1/36	1/6	1/12
$X = 0$	1/18	0	1/18	0
$X = 1$	0	1/36	1/6	1/12
$X = 2$	1/12	0	1/12	1/6

Table 1: Table 4.6: Joint distribution.

- (a) What is $P(X \geq 1 \text{ and } Y \leq 0)$?
- (b) What is the conditional probability that $Y \leq 0$ given that $X = 2$?
- (c) Are X and Y independent?
- (d) What is the distribution of $Z = XY$?

Solution 10.

- (a) This consists of the entries in the table where $X \geq 1$ and $Y \leq 0$, so we get

$$0 + \frac{1}{36} + \frac{1}{12} + 0 = \frac{1}{9}.$$

- (b) The key is the row of $X = 2$. We get

$$\frac{\frac{1}{12} + 0}{\frac{1}{12} + 0 + \frac{1}{12} + \frac{1}{6}} = \frac{1}{4}.$$

- (c) No. For example $P(X = 0, Y = 0) = 0$, which is not $P(X = 0)P(Y = 0)$ because

$$P(X = 0) = \frac{1}{18} + \frac{1}{18} = \frac{1}{9},$$

and

$$P(Y = 0) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}.$$

- (d) Z can take the values $-2, -1, 0, 1, 2, 4$. We have

$$P(Z = -2) = P(X = -1, Y = 2) + P(X = 2, Y = -1) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6},$$

$$\begin{aligned}
P(Z = -1) &= P(X = -1, Y = 1) + P(X = 1, Y = -1) = \frac{1}{6} + 0 = \frac{1}{6}, \\
P(Z = 1) &= P(X = -1, Y = -1) + P(X = 1, Y = 1) = 0 + \frac{1}{6} = \frac{1}{6}, \\
P(Z = 2) &= P(X = 1, Y = 2) + P(X = 2, Y = 1) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}, \\
P(Z = 4) &= P(X = 2, Y = 2) = \frac{1}{6}.
\end{aligned}$$

There are many ways $Z = 0$, but since it's the only case left, we can just use the complement and get

$$P(Z = 0) = 1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = \frac{1}{6}.$$

Therefore, Z is uniformly distributed!

2 Section 4.2

Problem 11. (Exercise 1)

Pick a point x at random (with uniform density) in the interval $[0, 1]$. Find the probability that $x > 1/2$, given that

- (a) $x > 1/4$.
- (b) $x < 3/4$.
- (c) $|x - 1/2| < 1/4$.
- (d) $x^2 - x + 2/9 < 0$.

Solution 11.

(a)

$$P\left(x > \frac{1}{2} \mid x > \frac{1}{4}\right) = \frac{P(x > \frac{1}{2})}{P(x > \frac{1}{4})} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}.$$

(b)

$$P\left(x > \frac{1}{2} \mid x < \frac{3}{4}\right) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

(c) We have

$$\left|x - \frac{1}{2}\right| < \frac{1}{4}$$

when

$$\frac{1}{4} < x < \frac{3}{4}.$$

Given that we want $x > 1/2$, we only have the range $1/2 < x < 3/4$, so we get that the probability is

$$\frac{\frac{1}{4}}{\frac{3}{4} - \frac{1}{4}} = \frac{1}{2}.$$

(d) We have

$$x^2 - x + \frac{2}{9} = \left(x - \frac{1}{2}\right)^2 - \frac{1}{36},$$

so the inequality is satisfied if

$$\left(x - \frac{1}{2}\right)^2 < \frac{1}{36},$$

which is equivalent to

$$\left| x - \frac{1}{2} \right| < \frac{1}{6}.$$

Since we want $x > 1/2$, we only get the interval from $1/2$ to $1/2 + 1/6$ which has length $1/6$. Therefore, the probability is

$$\frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{6}} = \frac{1}{2}.$$

Problem 12. (Exercise 2)

A radioactive material emits α -particles at a rate described by the density function

$$f(t) = .1e^{-.1t}.$$

Find the probability that a particle is emitted in the first 10 seconds, given that

- (a) no particle is emitted in the first second.
- (b) no particle is emitted in the first 5 seconds.
- (c) a particle is emitted in the first 3 seconds.
- (d) a particle is emitted in the first 20 seconds.

Solution 12.

- (a) We have $P(X > T) = e^{-0.1T}$, so

$$P(X \leq T) = 1 - e^{-0.1T}.$$

We want $P(X < 10 | X > 1)$, which is

$$\frac{P(1 < X < 10)}{P(X > 1)} = \frac{P(X < 10) - P(X < 1)}{P(X > 1)} = \frac{(1 - e^{-1}) - (1 - e^{-0.1})}{e^{-0.1}} = 1 - e^{-0.9} = 0.5934 \dots$$

Note that it is equivalent to finding the probability that a particle is emitted in the first 9 seconds.

- (b) This is equivalent to the probability that a particle is emitted in the first 5 seconds, so the answer is

$$1 - e^{-0.5} = 0.3935 \dots$$

- (c) 1 because if $X < 3$ then clearly $X < 10$.

- (d) This would be $P(X < 10 | X < 20)$, so we have

$$P(X < 10 | X < 20) = \frac{P(X < 10)}{P(X < 20)} = \frac{1 - e^{-1}}{1 - e^{-2}} = 0.3719 \dots$$

Problem 13. (Exercise 4)

Suppose you toss a dart at a circular target of radius 10 inches. Given that the dart lands in the upper half of the target, find the probability that

- (a) it lands in the right half of the target.
- (b) its distance from the center is less than 5 inches.
- (c) its distance from the center is greater than 5 inches.
- (d) it lands within 5 inches of the point $(0, 5)$.

Solution 13.

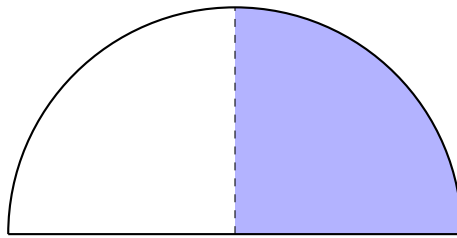


Figure 1: The event that the dart lands in the right half of the target.

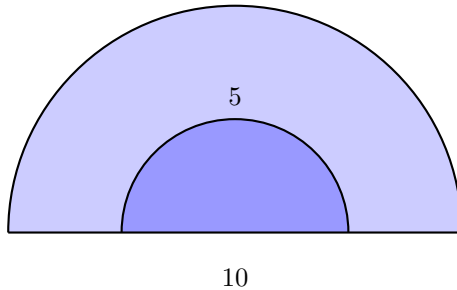


Figure 2: The event that the distance from the center is less than 5 inches.

(a) $1/2$.

(b)

$$\frac{5^2}{10^2} = \frac{1}{4}.$$

(c)

$$1 - \frac{1}{4} = \frac{3}{4}.$$

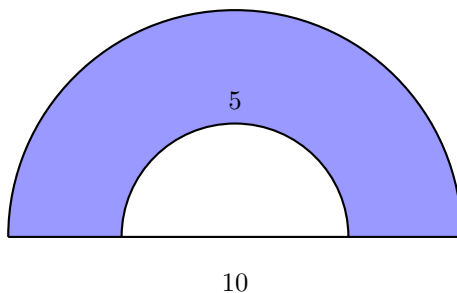


Figure 3: The event that the distance from the center is greater than 5 inches.

(d) The square of radius 5 centered at $(0,5)$ fits inside the dartboard and all of its area is in the upper half. Therefore, the probability is

$$\frac{5^2}{\frac{10^2}{2}} = \frac{1}{2}.$$

Problem 14. (Exercise 5)

Suppose you choose two numbers x and y , independently at random from the interval $[0, 1]$. Given that their sum lies in the interval $[0, 1]$, find the probability that

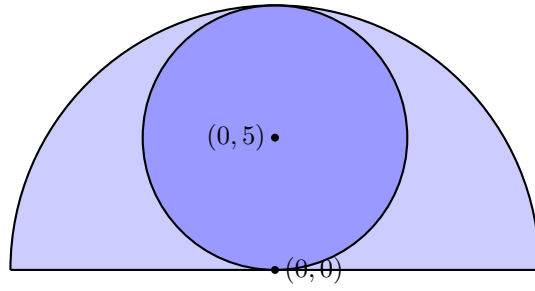


Figure 4: The event that the dart lands within 5 inches of $(0, 5)$.

- (a) $|x - y| < 1$.
- (b) $xy < 1/2$.
- (c) $\max\{x, y\} < 1/2$.
- (d) $x^2 + y^2 < 1/4$.
- (e) $x > y$.

Solution 14.

- (a) $|x - y| < 1$ is always satisfied, so the probability is 1.
- (b) If $x + y < 1$, then $xy < 1/2$. One can see it in the diagram, or one can show it with the AM-GM inequality, because

$$\sqrt{xy} \leq \frac{x + y}{2} \leq \frac{1}{2},$$

so

$$xy \leq \frac{1}{4} < \frac{1}{2}.$$

Therefore, the probability is 1.

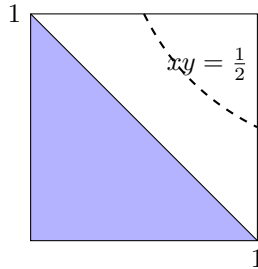


Figure 5: The event $xy < 1/2$.

- (c) The area of the region where $\{x, y\} < \frac{1}{2}$ is $\frac{1}{4}$ (and it is all inside the region where $x + y \leq 1$), while the area of the region where $x + y \leq 1$ is $\frac{1}{2}$. Therefore, the probability is $\frac{1}{2}$.
- (d) The region where $x + y \leq 1$ and $x^2 + y^2 < 1/4$ is a quarter of a circle of radius $1/2$, so the probability is

$$\frac{\frac{1}{4}\pi\left(\frac{1}{2}\right)^2}{\frac{1}{2}} = \frac{\pi}{8}.$$

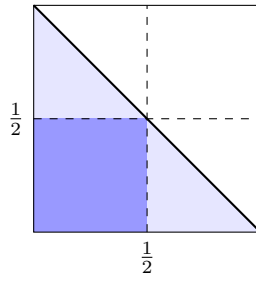


Figure 6: The event $\max\{x, y\} < 1/2$.

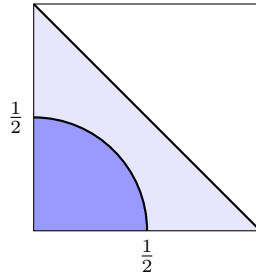


Figure 7: The event $x^2 + y^2 < 1/4$.

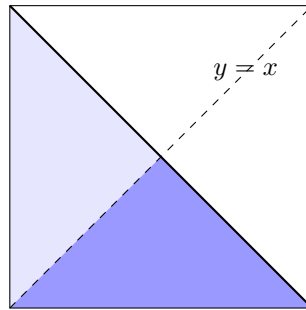


Figure 8: The event $x > y$.

(e) It's symmetric, so the probability is $1/2$, but one can also see it in the diagram.

Problem 15. (Exercise 10)

In Exercise 2.2.12 you proved the following: If you take a stick of unit length and break it into three pieces, choosing the breaks at random (i.e., choosing two real numbers independently and uniformly from $[0, 1]$), then the probability that the three pieces form a triangle is $1/4$. Consider now a similar experiment: First break the stick at random, then break the longer piece at random. Show that in the second experiment the theoretical probability of success is actually $2 \log 2 - 1$.

Solution 15. Let x be a random real number between 0 and 1. If $x < 1/2$, then let y be a random real number between 0 and $1 - x$. The three sticks will have lengths x, y and $1 - x - y$. For the sticks to form a triangle, we need

$$x + y > 1 - x - y,$$

$$x + (1 - x - y) > y,$$

and

$$y + (1 - x - y) > x.$$

In other words, we need to find the probability that

$$x + y > \frac{1}{2}, \quad y < \frac{1}{2}, \quad \text{and} \quad x < \frac{1}{2},$$

given that y is chosen uniformly from $[0, 1 - x]$. Looking at Figure 9, what we want is to integrate on the blue area and divide over integrating on the lighter blue area. Therefore, we have

$$\left(\frac{1}{2}\right) \frac{\int_0^{\frac{1}{2}} \int_{\frac{1}{2}-x}^{\frac{1}{2}} \frac{1}{1-x} dy dx}{\int_0^{\frac{1}{2}} \int_0^{1-x} \frac{1}{1-x} dy dx}.$$

Let's calculate each integral. The one on the denominator is

$$\int_0^{\frac{1}{2}} \int_0^{1-x} \frac{1}{1-x} dy dx = \int_0^{\frac{1}{2}} \frac{1-x}{1-x} dx = \frac{1}{2}.$$

This $1/2$ cancels the $1/2$ that was multiplying the expression, so the probability for the $x < 1/2$ case is just the integral of the numerator, which is

$$\int_0^{\frac{1}{2}} \int_{\frac{1}{2}-x}^{\frac{1}{2}} \frac{1}{1-x} dy dx = \int_0^{\frac{1}{2}} \frac{x}{1-x} dx = \int_0^{\frac{1}{2}} \left(\frac{1}{1-x} - 1 \right) dx = (-\log(1-x) - x) \Big|_0^{\frac{1}{2}} = \log 2 - \frac{1}{2}.$$

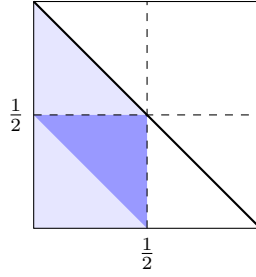


Figure 9: The case when $x < 1/2$.

The case $x > 1/2$ must be symmetric, so the answer is

$$2 \log 2 - 1.$$

Just for fun, we can analyze the case in all its analytical glory:. For $x > 1/2$, then let y be a random real number between 0 and x . In this case, the three sticks have lengths y , $x - y$, and $1 - x$. For the sticks to form a triangle, we need

$$(x - y) + y > 1 - x,$$

$$(x - y) + (1 - x) > y,$$

and

$$y + (1 - x) > x - y.$$

In other words, we need to find the probability that

$$x - y < \frac{1}{2}, \quad y < \frac{1}{2}, \quad \text{and} \quad x > \frac{1}{2}.$$

We need to integrate $\frac{1}{x}$ in the colored region from Figure 10 and divide over the integral over the lighter shaded region in Figure 10. After multiplying by $1/2$, that will be the contribution to the probability for this side.

Therefore, the probability for this case is

$$\left(\frac{1}{2}\right) \frac{\int_{\frac{1}{2}}^1 \int_{-\frac{1}{2}+x}^{\frac{1}{2}} \frac{1}{x} dy dx}{\int_{\frac{1}{2}}^1 \int_0^x \frac{1}{x} dy dx}.$$

As before, let's calculate the bottom integral first:

$$\int_{\frac{1}{2}}^1 \int_0^x \frac{1}{x} dy dx = \int_{\frac{1}{2}}^1 1 dx = \frac{1}{2}.$$

This 1/2 cancels the other 1/2, so the probability is just the integral in the numerator. The integral in the numerator is

$$\int_{\frac{1}{2}}^1 \int_{-\frac{1}{2}+x}^{\frac{1}{2}} \frac{1}{x} dy dx = \int_{\frac{1}{2}}^1 \frac{1-x}{x} dx = (\log x - x) \Big|_{\frac{1}{2}}^1 = \log 2 - \frac{1}{2}.$$

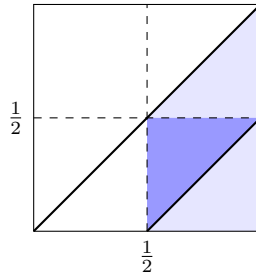


Figure 10: The case when $x > 1/2$.