

Homework 4 Solutions

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1 Section 5.1

Problem 1. (Exercise 1)

For which of the following random variables would it be appropriate to assign a uniform distribution?

- (a) Let X represent the roll of one die.
- (b) Let X represent the number of heads obtained in three tosses of a coin.
- (c) A roulette wheel has 38 possible outcomes: 0, 00, and 1 through 36. Let X represent the outcome when a roulette wheel is spun.
- (d) Let X represent the birthday of a randomly chosen person.
- (e) Let X represent the number of tosses of a coin necessary to achieve a head for the first time.

Solution 1.

- (a) Yes.
- (b) No.
- (c) Yes.
- (d) Sort of. It's a good estimate.
- (e) No.

Problem 2. (Exercise 4)

Suppose we are attending a college which has 3000 students. We wish to choose a subset of size 100 from the student body. Let X represent the subset, chosen using the following possible strategies. For which strategies would it be appropriate to assign the uniform distribution to X ? If it is appropriate, what probability should we assign to each outcome?

- (a) Take the first 100 students who enter the cafeteria to eat lunch.
- (b) Ask the Registrar to sort the students by their Social Security number, and then take the first 100 in the resulting list.
- (c) Ask the Registrar for a set of cards, with each card containing the name of exactly one student, and with each student appearing on exactly one card. Throw the cards out of a third-story window, then walk outside and pick up the first 100 cards that you find.

Solution 2.

- (a) Not appropriate because not everybody goes to the cafeteria, so the probabilities are not equally likely.

- (b) Not appropriate because Social Security numbers before 2011 included geographical information, which makes their sorting not random. If all students got their social security number after 2011, then it would be random enough to suggest uniform distribution where each subset has probability

$$\frac{1}{\binom{3000}{100}}.$$

- (c) Appropriate. The probability for each subset would be

$$\frac{1}{\binom{3000}{100}}.$$

Problem 3. (Exercise 6)

Let $X_1, X_2, \dots, X_n$ be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k . Let Y denote the minimum of the X_i 's. Find the distribution of Y .

Solution 3. Let $m \in \{0, 1, 2, \dots, k\}$. Then

$$P(Y > m) = \left(\frac{k-m}{k} \right)^n,$$

because if $Y > m$ it means $X_1, X_2, \dots, X_n$ are all greater than m and each one is independent, so the probability they are all greater than m is $\left(\frac{k-m}{k} \right)^n$.

Therefore, for $m \in \{1, 2, \dots, k\}$,

$$P(Y = m) = P(Y > m-1) - P(Y > m) = \left(\frac{k-m+1}{k} \right)^n - \left(\frac{k-m}{k} \right)^n.$$

Problem 4. (Exercise 7)

A die is rolled until the first time T that a six turns up.

- (a) What is the probability distribution for T ?
 (b) Find $P(T > 3)$.
 (c) Find $P(T > 6 | T > 3)$.

Solution 4.

- (a)

$$P(T = k) = \left(\frac{5}{6} \right)^{k-1} \left(\frac{1}{6} \right).$$

- (b) For $T > 3$ we need the first three dice rolls to not be 6, therefore

$$P(T > 3) = \left(\frac{5}{6} \right)^3.$$

- (c) Similarly as in (b), we have

$$P(T > 6) = \left(\frac{5}{6} \right)^6,$$

so

$$P(T > 6 | T > 3) = \frac{\left(\frac{5}{6} \right)^6}{\left(\frac{5}{6} \right)^3} = \left(\frac{5}{6} \right)^3.$$

Problem 5. (Exercise 16)

Assume that, during each second, a Dartmouth switchboard receives one call with probability .01 and no calls with probability .99. Use the Poisson approximation to estimate the probability that the operator will miss at most one call if she takes a 5-minute coffee break.

Solution 5. The expected number of calls in 5 minutes are

$$0.01 \times 300 = 3.$$

Therefore, the probability that the operator will miss at most one call is

$$P(X = 0) + P(X = 1) = e^{-3} + 3e^{-3} = \frac{4}{e^3} \approx 0.1991.$$

Problem 6. (Exercise 18)

A baker blends 600 raisins and 400 chocolate chips into a dough mix and, from this, makes 500 cookies.

- Find the probability that a randomly picked cookie will have no raisins.
- Find the probability that a randomly picked cookie will have exactly two chocolate chips.
- Find the probability that a randomly chosen cookie will have at least two bits (raisins or chips) in it.

Solution 6.

- Let's call the cookies $C_1, C_2, \dots, C_{500}$. We can model the raisin distribution by choosing a cookie for each raisin (there are 500 choices each time). The probability that the randomly picked cookie does not have a raisin is therefore

$$\left(\frac{499}{500}\right)^{600} \approx 0.300833.$$

That is because, each raisin had to go to one of the other cookies, so the raisin had 499 out of 500 possibilities and that's for each raisin.

Note: One can get a good approximation using Poisson with $\lambda = \frac{600}{500} = 1.2$. Indeed, Poisson suggests the answer is

$$e^{-1.2} \approx 0.301194.$$

- There are two chocolate chips that chose the same cookie (there are $\binom{400}{2}$ ways of choosing the chocolate chips), the rest picked one of the other 499 cookies (each one had 499 out of 500 choices). Therefore,

$$\binom{400}{2} \left(\frac{1}{500}\right)^2 \left(\frac{499}{500}\right)^{398} \approx 0.143886.$$

Note: One can get a good approximation using Poisson with $\lambda = \frac{400}{500} = 0.8$. Indeed, Poisson suggests the answer is

$$\frac{\lambda^2 e^{-\lambda}}{2} \approx 0.143785.$$

- We can consider the 1000 raisins and chocolate chips as the bits. Reasoning as before we can calculate the probability of 0 bits and of 1 bit and then use the complement technique to get

$$1 - \left(\frac{499}{500}\right)^{1000} - \binom{1000}{1} \left(\frac{1}{500}\right)^1 \left(\frac{499}{500}\right)^{999} \approx 0.594265.$$

Note: One can get a good approximation using Poisson with $\lambda = \frac{1000}{500} = 2$. Indeed, Poisson suggests the answer is

$$1 - e^{-2} - 2e^{-2} = 1 - \frac{3}{e^2} \approx 0.593994.$$

Problem 7. (Exercise 25)

Reese Prosser never puts money in a 10-cent parking meter in Hanover. He assumes that there is a probability of .05 that he will be caught. The first offense costs nothing, the second costs 2 dollars, and subsequent offenses cost 5 dollars each. Under his assumptions, how does the expected cost of parking 100 times without paying the meter compare with the cost of paying the meter each time?

Solution 7. Let n be the number of times he gets ticketed. If $n \geq 2$, then the fine amount is $2 + 5(n - 2) = 5n - 8$ (because he gets fines \$2 for the second offence and \$5 for each one after). Therefore, the expected cost is

$$\begin{aligned} \sum_{n=2}^{100} (5n - 8) \binom{100}{n} (0.05)^n (0.95)^{100-n} &= 5 \sum_{n=2}^{100} n \binom{100}{n} (0.05)^n (0.95)^{100-n} - 8 \sum_{n=2}^{100} \binom{100}{n} (0.05)^n (0.95)^{100-n} \\ &= 5 \left(5 - \binom{100}{1} (0.05)(0.95)^{99} \right) - 8 \left(1 - \binom{100}{1} (0.05)(0.95)^{99} - (0.95)^{100} \right) \\ &= 17 + 300(0.05)(0.95)^{99} + 8(0.95)^{100} = 17.1408. \end{aligned}$$

Note that it is bigger than \$10, so it is not a good strategy.

Note: If one approximates the answer using the Poisson distribution, one would get

$$\sum_{n=2}^{\infty} (5n - 8) \frac{5^n}{n!} e^{-5} = 17 + 23e^{-5} = 17.155.$$

Problem 8. (Exercise 29)

The king's coinmaster boxes his coins 500 to a box and puts 1 counterfeit coin in each box. The king is suspicious, but, instead of testing all the coins in 1 box, he tests 1 coin chosen at random out of each of 500 boxes. What is the probability that he finds at least one fake? What is it if the king tests 2 coins from each of 250 boxes?

Solution 8. For the example with 500 boxes, the answer is

$$1 - \left(\frac{499}{500} \right)^{500},$$

for the example with 250 boxes, the answer is

$$1 - \left(\frac{249}{250} \right)^{250}.$$

Note: Both can be approximated using Poisson distribution by

$$1 - e^{-1}.$$

Problem 9. (Exercise 30)

(From Kemeny) Show that, if you make 100 bets on the number 17 at roulette at Monte Carlo (see Example 6.13), you will have a probability greater than 1/2 of coming out ahead. What is your expected winning?

Solution 9. To come out ahead, you need to win at least 3 bets in 100 tries. That probability is

$$1 - \binom{100}{2} \left(\frac{1}{37} \right)^2 \left(\frac{36}{37} \right)^{98} - \binom{100}{1} \left(\frac{1}{37} \right)^1 \left(\frac{36}{37} \right)^{99} - \left(\frac{36}{37} \right)^{100} = 0.509394 \dots > \frac{1}{2}.$$

The expected value for 100 bets is

$$100 \left(35 \cdot \left(\frac{1}{37} \right) + (-1) \left(\frac{36}{37} \right) \right) = -\frac{100}{37}.$$

2 Section 6.1

Problem 10. (Exercise 1)

A card is drawn at random from a deck consisting of cards numbered 2 through 10. A player wins 1 dollar if the number on the card is odd and loses 1 dollar if the number is even. What is the expected value of his winnings?

Solution 10. The probability that the number is odd is $4/9$, while the probability that it is even is $5/9$, therefore the expected value is

$$\frac{4}{9}(1) + \frac{5}{9}(-1) = -\frac{1}{9}.$$

Problem 11. (Exercise 3)

In a class there are 20 students: 3 are 5' 6", 5 are 5' 8", 4 are 5' 10", 4 are 6', and 4 are 6' 2". A student is chosen at random. What is the student's expected height?

Solution 11. We just need to average the 20 heights. Let's translate to inches first: 3 are 66", 5 are 68", 4 are 70", 4 are 72", and 4 are 74", so we have

$$\frac{3 \times 66 + 5 \times 68 + 4 \times 70 + 4 \times 72 + 4 \times 74}{20} = \frac{1402}{20} = 70.1''.$$

Problem 12. (Exercise 6)

A die is rolled twice. Let X denote the sum of the two numbers that turn up, and Y the difference of the numbers (specifically, the number on the first roll minus the number on the second). Show that $E(XY) = E(X)E(Y)$. Are X and Y independent?

Solution 12. Let X_1 be the outcome of the first dice and X_2 of the second dice. Suppose $X_1 = a$ and $X_2 = b$, then $X = a + b$ and $Y = a - b$, so $XY = a^2 - b^2$. Therefore

$$E[XY] = E[X_1^2] - E[X_2^2] = 0.$$

We also know

$$E[Y] = E[X_1] - E[X_2] = 0.$$

The events are not independent, because X and Y have the same parity, so the probability that $X = 4$ and $Y = 1$ is 0, yet $P(X = 4) \neq 0$ and $P(Y = 1) \neq 0$. Therefore

$$P(X = 4, Y = 1) \neq P(X = 4)P(Y = 1).$$

Problem 13. (Exercise 13)

You have 80 dollars and play the following game. An urn contains two white balls and two black balls. You draw the balls out one at a time without replacement until all the balls are gone. On each draw, you bet half of your present fortune that you will draw a white ball. What is your expected final fortune?

Solution 13. There are six possibilities. Let's find the payout on each of them:

Possibility	Payout
WWBB	$80 \rightarrow 120 \rightarrow 180 \rightarrow 90 \rightarrow 45$
WBWB	$80 \rightarrow 120 \rightarrow 60 \rightarrow 90 \rightarrow 45$
WBBW	$80 \rightarrow 120 \rightarrow 60 \rightarrow 30 \rightarrow 45$
BWBB	$80 \rightarrow 40 \rightarrow 60 \rightarrow 90 \rightarrow 45$
BWBW	$80 \rightarrow 40 \rightarrow 60 \rightarrow 30 \rightarrow 45$
BBWW	$80 \rightarrow 40 \rightarrow 20 \rightarrow 30 \rightarrow 45$

All paths lead to 45.

Another way to calculate is to note that when you win you multiply your money by $\frac{3}{2}$ and when you lose by $\frac{1}{2}$, so the overall outcome should be

$$\frac{3^2}{2^4}(80) = 45.$$

Problem 14. (Exercise 14)

In the hat check problem (see Example 3.12), it was assumed that N people check their hats and the hats are handed back at random. Let $X_j = 1$ if the j th person gets his or her hat and 0 otherwise. Find $E(X_j)$ and $E(X_j \cdot X_k)$ for j not equal to k . Are X_j and X_k independent?

Solution 14. Person j can get any of the N hats, so $E(X_j) = \frac{1}{N}$. It can also be calculated as the ratio of

$$\frac{(N-1)!}{N!} = \frac{1}{N},$$

where the ratio comes from the $(N-1)!$ factorial ways that person j gets their hat back.

If j and k get their hat back, there are $(N-2)!$ ways of arranging the other $N-2$ hats. Therefore

$$E(X_j \cdot X_k) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}.$$

Since it is not $E(X_j)E(X_k)$, then X_j and X_k are not independent.

Problem 15. (Exercise 15)

A box contains two gold balls and three silver balls. You are allowed to choose successively balls from the box at random. You win 1 dollar each time you draw a gold ball and lose 1 dollar each time you draw a silver ball. After a draw, the ball is not replaced. Show that, if you draw until you are ahead by 1 dollar or until there are no more gold balls, this is a favorable game.

Solution 15. The times that we lose, we lose \$1. The times that we win, we win \$1.

The times that we lose happen when the fifth ball is gold, and the first ball is silver. Therefore, the probability of such an event is

$$\binom{3}{2} \frac{3! \cdot 2!}{5!} = \frac{3}{10}.$$

The times that we tie involve when there have been two gold balls and two silver balls where the first ball is silver and the last ball is gold. The probability of such an event is

$$\binom{2}{1} \frac{3 \cdot 2 \cdot 2 \cdot 1}{5!} = \frac{1}{5}.$$

Therefore, the rest of the time we win, so we win with probability $\frac{1}{2}$.

It follows that the expected value is

$$\frac{1}{2} - \frac{3}{10} = \frac{1}{5}.$$

Note: One can also do the casework of analysing all scenarios: G, SGG, SGSG, SGSSG, SSGG, SSGSG. The probability of G is $\frac{2}{5}$, the probability of SGG is $\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{10}$, and so on.

Problem 16. (Exercise 18)

Exactly one of six similar keys opens a certain door. If you try the keys, one after another, what is the expected number of keys that you will have to try before success?

Solution 16. Let X be the random variable that takes on as a value the number of keys tries to be successful. Then $P(X = i) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$ (because the correct key can be in any of the six locations with equal likelihood). Therefore

$$E(X) = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}.$$