

Homework 5 Solutions

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1 Section 5.2

Problem 1. (Exercise 1)

Choose a number U from the unit interval $[0, 1]$ with uniform distribution. Find the cumulative distribution and density for the random variables

(a) $Y = U + 2$.

(b) $Y = U^3$.

Solution 1.

(a) The density function is

$$f(x) = \begin{cases} 1 & \text{if } 2 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

The cumulative distribution is

$$F(x) = \begin{cases} 0 & \text{if } x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 3 \\ 1 & \text{if } x > 3. \end{cases}$$

(b) The cumulative distribution is

$$F(x) = P(U^3 \leq x) = P(U \leq x^{1/3}) = \begin{cases} 0 & \text{if } x < 0 \\ x^{1/3} & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

Taking the derivative we get the density function:

$$f(x) = \begin{cases} \frac{1}{3}x^{-\frac{2}{3}} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2. (Exercise 2)

Choose a number U from the interval $[0, 1]$ with uniform distribution. Find the cumulative distribution and density for the random variables

(a) $Y = 1/(U + 1)$.

(b) $Y = \log(U + 1)$.

Solution 2.

- (a) Let's get the cumulative distribution first by finding $P(Y \leq x)$.

We have for $x > 0$,

$$F(x) = P(Y \leq x) = P\left(\frac{1}{U+1} \leq x\right) = P\left(U \geq \frac{1}{x} - 1\right).$$

Note that for $x > 1$, $\frac{1}{x} - 1$ is negative, so the probability that U is greater than that is 1. For $x \leq 0$, by definition $F(x) = 0$ as well. We may assume $0 < x \leq 1$. When $\frac{1}{x} - 1 \geq 1$, then $F(x) = 0$. That happens for $x \leq \frac{1}{2}$. Therefore, the only place where we have a value is when $0 \leq \frac{1}{x} - 1 \leq 1$. But in those cases the cumulative distribution function is

$$1 - \left(\frac{1}{x} - 1\right) = 2 - \frac{1}{x}.$$

Therefore

$$F(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ 2 - \frac{1}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

We can find the density function by taking the derivative. We get

$$f(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ \frac{1}{x^2} & \text{if } \frac{1}{2} \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}.$$

- (b) Assume $0 \leq x \leq 1$. We want to find $F(x)$. We want to figure out the probability that $\log(U+1) \leq x$. But that's equivalent to $U \leq e^x - 1$. when $x > \log 2$ we have $e^x - 1 > 1$, so in those cases $F(x) = 1$. Therefore, we can assume $0 < x < \log 2$. For those, we get $F(x) = e^x - 1$. Then

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^x - 1 & \text{if } 0 \leq x \leq \log 2 \\ 1 & \text{if } x > \log 2 \end{cases}.$$

We can find the density function by taking the derivative. We get

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq \log 2 \\ 0 & \text{if } x > \log 2 \end{cases}.$$

Problem 3. (Exercise 7)

Explain how you can generate a random variable whose cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x^2, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Solution 3. We want some random variable X such that $P(X \leq x)$ is x^2 when $0 \leq x \leq 1$. Let Y be uniformly distributed from the interval $[0, 1]$, then $X = \sqrt{Y}$ satisfies, for $0 \leq x \leq 1$, that

$$P(X \leq x) = P(\sqrt{Y} \leq x) = P(Y \leq x^2) = x^2.$$

Problem 4. (Exercise 10)

Let U, V be random numbers chosen independently from the interval $[0, 1]$. Find the cumulative distribution and density for the random variables

(a) $Y = \max(U, V)$.

(b) $Y = \min(U, V)$.

Solution 4.

- (a) We want the probability that $\max(U, V) \leq x$ when $0 \leq x \leq 1$. Since $\max(U, V) \leq x$, then $U \leq x$ and $V \leq x$. It follows that the probability is x^2 . Therefore

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x^2, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

The density function is thus

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

- (b) The probability that $\min(U, V) \leq x$ is $2x - x^2$, because it's

$$P((U \leq x) \cup (V \leq x)) = P(U \leq x) + P(V \leq x) - P((U \leq x) \cap (V \leq x)) = x + x - x^2 = 2x - x^2.$$

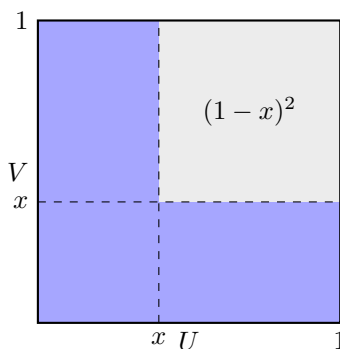


Figure 1: The event $\min(U, V) \leq x$.

It follows that

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 2x - x^2, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

The density function is thus

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 2 - 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Problem 5. (Exercise 16)

Let X be a random variable with density function

$$f_X(x) = \begin{cases} cx(1-x), & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) What is the value of c ?

(b) What is the cumulative distribution function F_X for X ?

(c) What is the probability that $X < 1/4$?

Solution 5.

(a) For it to be a density function the total integral must be one, therefore we have to solve

$$\int_0^1 cx(1-x) dx = 1.$$

We have

$$\begin{aligned}\int_0^1 cx(1-x) dx &= c \int_0^1 x - x^2 dx = c \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 \\ &= c \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{c}{6}.\end{aligned}$$

Therefore, $c = 6$.

(b) We can get it from integrating, so we have

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 3x^2 - 2x^3 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

(c) That would be $F(1/4)$, so that is

$$F\left(\frac{1}{4}\right) = 3\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right)^3 = \frac{3}{16} - \frac{2}{64} = \frac{5}{32}.$$

Problem 6. (Exercise 25)

Let X be a random variable normally distributed with parameters $\mu = 70$, $\sigma = 10$. Estimate

(a) $P(X > 50)$.

(b) $P(X < 60)$.

(c) $P(X > 90)$.

(d) $P(60 < X < 80)$.

Solution 6.

(a) The z -score is -2 . Therefore

$$P(X > 50) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{\infty} e^{-x^2/2} dx \approx 0.97725.$$

Note: One can also do it directly using the formula in terms of μ and σ getting

$$\frac{1}{10\sqrt{2\pi}} \int_{50}^{\infty} e^{-\frac{(x-70)^2}{200}} dx.$$

(b) The z -score is -1 . Therefore

$$P(X < 60) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-x^2/2} dx \approx 0.158655.$$

(c) The z -score is 2. Therefore

$$P(X > 90) = \frac{1}{\sqrt{2\pi}} \int_2^{\infty} e^{-x^2/2} dx \approx 0.0227501.$$

(d) We want the probability for z to be between -1 and 1, so

$$P(60 < X < 80) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-x^2/2} dx \approx 0.682689.$$

Problem 7. (Exercise 28)

(Ross) An expert witness in a paternity suit testifies that the length (in days) of a pregnancy, from conception to delivery, is approximately normally distributed, with parameters $\mu = 270$, $\sigma = 10$. The defendant in the suit is able to prove that he was out of the country during the period from 290 to 240 days before the birth of the child. What is the probability that the defendant was in the country when the child was conceived?

Solution 7. The z -score for 240 is -3 and the z -score for 290 is 2. Therefore, we are looking for

$$1 - P(-3 < z < 2) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-3}^2 e^{-x^2/2} dx \approx 0.0241.$$

Problem 8. (Exercise 29)

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is the probability that the repair time exceeds 4 hours? If it exceeds 4 hours what is the probability that it exceeds 8 hours?

Solution 8. The probability that it exceeds 4 hours is

$$e^{-\frac{1}{2} \cdot 4} = e^{-2} \approx 0.135335.$$

The exponential distribution is memoryless, so it should be the same probability. Therefore, it is also

$$e^{-2} \approx 0.135335.$$

2 Section 6.2

Problem 9. (Exercise 1)

A number is chosen at random from the set $S = \{-1, 0, 1\}$. Let X be the number chosen. Find the expected value, variance, and standard deviation of X .

Solution 9. We have

$$E(X) = \frac{1}{3}(-1) + \frac{1}{3}(0) + \frac{1}{3}(1) = 0.$$

Therefore

$$V(X) = E(X^2) = \frac{1}{3}(-1)^2 + \frac{1}{3}(0)^2 + \frac{1}{3}(1)^2 = \frac{2}{3}.$$

Then

$$D(X) = \sqrt{\frac{2}{3}}.$$

Problem 10. (Exercise 2)

A random variable X has the distribution

$$p_X = \begin{pmatrix} 0 & 1 & 2 & 4 \\ 1/3 & 1/3 & 1/6 & 1/6 \end{pmatrix}.$$

Find the expected value, variance, and standard deviation of X .

Solution 10. We have

$$E(X) = \frac{1}{3}(0) + \frac{1}{3}(1) + \frac{1}{6}(2) + \frac{1}{6}(4) = \frac{4}{3}.$$

Therefore

$$V(X) = E(X^2) - \left(\frac{4}{3}\right)^2 = \frac{1}{3}(0)^2 + \frac{1}{3}(1)^2 + \frac{1}{6}(2)^2 + \frac{1}{6}(4)^2 - \frac{16}{9} = \frac{17}{9}.$$

Then

$$D(X) = \frac{\sqrt{17}}{3}.$$

Problem 11. (Exercise 4)

X is a random variable with $E(X) = 100$ and $V(X) = 15$. Find

- (a) $E(X^2)$.
- (b) $E(3X + 10)$.
- (c) $E(-X)$.
- (d) $V(-X)$.
- (e) $D(-X)$.

Solution 11.

(a)

$$V(X) = E(X^2) - (E(X))^2,$$

so

$$E(X^2) = V(X) + (E(X))^2 = 15 + 100^2 = 10015.$$

(b) By linearity of expectation, we have

$$E(3X + 10) = 3E(X) + 10 = 310.$$

(c) By linearity of expectation

$$E(-X) = -E(X) = -100.$$

(d) We have

$$V(-X) = (-1)^2 V(X) = 15.$$

(e) Therefore

$$D(-X) = \sqrt{15}.$$

Problem 12. (Exercise 8)

A random sample of 2400 people are asked if they favor a government proposal to develop new nuclear power plants. If 40 percent of the people in the country are in favor of this proposal, find the expected value and the standard deviation for the number S_{2400} of people in the sample who favored the proposal.

Solution 12. We have $p = 0.4$ and $n = 2400$. Let $X = S_{2400}$. The expected value of X is np , so

$$E(X) = 0.4(2400) = 960.$$

The variance is npq , so

$$V(X) = 0.4(2400)(0.6) = 576.$$

Therefore, the standard deviation is

$$\sqrt{576} = 24.$$

Problem 13. (Exercise 15)

Suppose that n people have their hats returned at random. Let $X_i = 1$ if the i th person gets his or her own hat back and 0 otherwise. Let

$$S_n = \sum_{i=1}^n X_i.$$

Then S_n is the total number of people who get their own hats back. Show that

- (a) $E(X_i^2) = 1/n$.
- (b) $E(X_i X_j) = 1/n(n-1)$ for $i \neq j$.
- (c) $E(S_n^2) = 2$ (using (a) and (b)).
- (d) $V(S_n) = 1$.

Solution 13. The problem as written requires assuming $n \geq 2$ for (b), (c), and (d), so in what follows we will assume $n \geq 2$.

- (a) X_i is 1 with probability $\frac{1}{n}$ and 0 with probability $\frac{n-1}{n}$, therefore

$$E(X_i^2) = \frac{1}{n}1^2 + \frac{n-1}{n}0^2 = \frac{1}{n}.$$

- (b) There are $(n-2)!$ permutations where i and j get their hat back, so

$$E(X_i X_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

- (c) We have

$$E(S_n^2) = E((X_1 + X_2 + \cdots + X_n)^2) = \sum_{i=1}^n E(X_i^2) + 2 \sum_{i=1}^n \sum_{j=i+1}^n E(X_i X_j).$$

The last equality comes from noticing that every product of two terms appears with a constant 2 in front of it. But from (a) and (b) we can now get

$$\begin{aligned} E(S_n^2) &= \sum_{i=1}^n \frac{1}{n} + 2 \sum_{i=1}^n \sum_{j=i+1}^n \frac{1}{n(n-1)} \\ &= 1 + \frac{2}{n(n-1)} \binom{n}{2} = 1 + 1 = 2. \end{aligned}$$

- (d) We have $E(S_n) = 1$ from linearity of expectation, so then

$$V(S_n) = E(S_n^2) - (E(S_n))^2 = 2 - 1 = 1.$$

3 Section 6.3

Problem 14. (Exercise 1)

Let X be a random variable with range $[-1, 1]$ and let $f_X(x)$ be the density function of X . Find $\mu(X)$ and $\sigma^2(X)$ if, for $|x| < 1$,

- (a) $f_X(x) = 1/2$.
- (b) $f_X(x) = |x|$.
- (c) $f_X(x) = 1 - |x|$.

(d) $f_X(x) = (3/2)x^2$.

Solution 14.

(a) $\mu = 0$ by symmetry. Then

$$\sigma^2 = E(X^2) = \int_{-1}^1 \frac{1}{2}x^2 dx = \frac{x^3}{6} \Big|_{-1}^1 = \frac{1}{3}.$$

(b) We have

$$\mu = \int_{-1}^1 x \cdot |x| dx = 0.$$

Then

$$\sigma^2 = E(X^2) = \int_{-1}^1 x^2|x| dx = 2 \int_0^1 x^3 dx = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

(c) We have

$$\mu = \int_{-1}^1 (1 - |x|)x dx = \int_{-1}^1 x dx - \int_{-1}^1 x|x| dx = 0 - 0 = 0.$$

Then

$$\sigma^2 = E(X^2) = \int_{-1}^1 (1 - |x|)x^2 dx = \int_{-1}^1 x^2 dx - \int_{-1}^1 |x|x^2 dx = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

(d) We have

$$\mu = \int_{-1}^1 \frac{3x^2}{2} \cdot x dx = \frac{3}{2} \int_{-1}^1 x^3 dx = 0.$$

Then

$$\sigma^2 = E(X^2) = \int_{-1}^1 \frac{3x^2}{2} \cdot x^2 dx = 3 \int_0^1 x^4 dx = \frac{3}{5}.$$

Problem 15. (Exercise 5)

Let X be a random variable with range $[-1, 1]$ and density function $f_X(x) = ax^2 + bx + c$ if $|x| < 1$ and 0 otherwise.

(a) Show that $\frac{2a}{3} + 2c = 1$ (see Exercise 4).

(b) Show that $\frac{2b}{3} = \mu(X)$.

(c) Show that $\frac{2a}{5} + \frac{2c}{3} - \frac{4b^2}{9} = \sigma^2(X)$.

(d) Find a , b , and c if $\mu(X) = 0$, $\sigma^2(X) = 1/15$, and sketch the graph of f_X .

(e) Find a , b , and c if $\mu(X) = 0$, $\sigma^2(X) = 1/2$, and sketch the graph of f_X .

Solution 15.

(a) We have that the integral of f between -1 and 1 must be 1. But

$$\int_{-1}^1 ax^2 + bx + c dx = 2a \int_0^1 x^2 dx + b \int_{-1}^1 x dx + 2c = \frac{2a}{3} + 2c.$$

Therefore

$$\frac{2a}{3} + 2c = 1.$$

(b) We have

$$\mu = \int_{-1}^1 x(ax^2 + bx + c) dx = a \int_{-1}^1 x^3 dx + b \int_{-1}^1 x^2 dx + c \int_{-1}^1 x dx = 0 + \frac{2b}{3} + 0 = \frac{2b}{3}.$$

(c) We have

$$\sigma^2 = \left(\int_{-1}^1 x^2(ax^2 + bx + c) dx \right) - \left(\frac{2b}{3} \right)^2 = \frac{2a}{5} + \frac{2c}{3} - \frac{4b^2}{9}.$$

(d) From $\mu = 0$ we get $b = 0$. Then we have

$$\frac{2a}{3} + 2c = 1, \tag{1}$$

and

$$\frac{2a}{5} + \frac{2c}{3} = \sigma^2 = \frac{1}{15}.$$

Therefore

$$\frac{6a}{5} + 2c = \frac{1}{5}. \tag{2}$$

By subtracting (1) from (2), we get

$$\frac{8}{15}a = -\frac{4}{5},$$

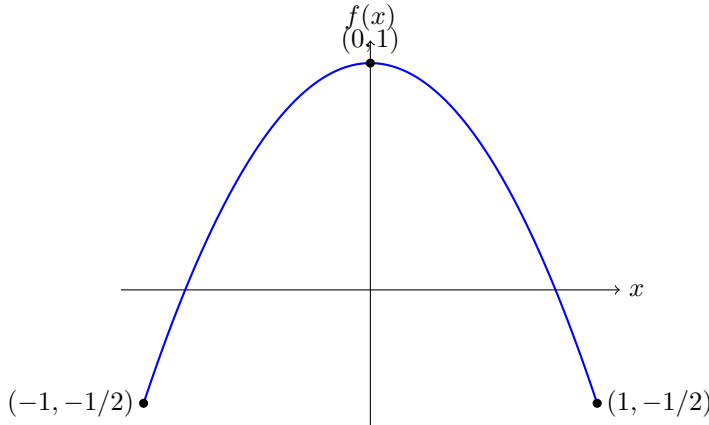
so

$$a = -\frac{3}{2}.$$

It follows that

$$c = 1.$$

The plot is below. Note that f is negative at some values, which contradicts that it is a probability density function. Therefore, there are no values of a, b, c for which a probability density function of that shape produces $\mu = 0$ and $\sigma^2 = \frac{1}{15}$.



(e) Similarly $b = 0$. We again have (1). From the variance, we have

$$\frac{1}{2} = \frac{2a}{5} + \frac{2c}{3}.$$

Therefore

$$\frac{3}{2} = \frac{6a}{5} + 2c. \tag{3}$$

Subtracting (1) from (3) we get

$$\frac{8}{15}a = \frac{1}{2},$$

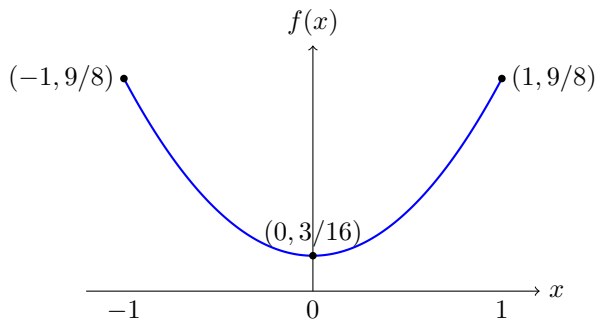
so

$$a = \frac{15}{16}.$$

It follows that

$$c = \frac{3}{16}.$$

This one is a possible density function and its plot is below.



Problem 16. (Exercise 6)

Let T be a random variable with range $[0, \infty]$ and f_T its density function. Find $\mu(T)$ and $\sigma^2(T)$ if, for $t < 0$, $f_T(t) = 0$, and for $t > 0$,

- (a) $f_T(t) = 3e^{-3t}$.
- (b) $f_T(t) = 9te^{-3t}$.
- (c) $f_T(t) = 3/(1+t)^4$.

Solution 16.

- (a) This is an exponential distribution with $\lambda = 3$, so

$$\mu = \frac{1}{\lambda} = \frac{1}{3},$$

and

$$\sigma^2 = \frac{1}{\lambda^2} = \frac{1}{9}.$$

- (b) We have

$$\mu = \int_0^{\infty} 9t^2 e^{-3t} dt.$$

To evaluate this integral we need to do integration by parts twice. We get

$$\mu = \left(-\frac{e^{-3t}}{3} (9t^2 + 6t + 2) \right) \Big|_0^{\infty} = \frac{2}{3}.$$

For σ^2 we need to get $E(X^2)$ first, which is

$$E(X^2) = \int_0^{\infty} 9t^3 e^{-3t} dt.$$

Doing integration by parts three times yields

$$E(X^2) = \left(-\frac{e^{-3t}}{3} (2 + 6t + 9t^2 + 9t^3) \right) \Big|_0^{\infty} = \frac{2}{3}.$$

Therefore

$$\sigma^2 = E(X^2) - \mu^2 = \frac{2}{3} - \left(\frac{2}{3} \right)^2 = \frac{2}{9}.$$

(c) We have

$$\mu = \int_0^\infty \frac{3t}{(1+t)^4} dt.$$

Change the variable by using $u = 1 + t$. Then

$$\begin{aligned}\mu &= 3 \int_1^\infty \frac{u-1}{u^4} du = 3 \int_1^\infty u^{-3} du - 3 \int_1^\infty u^{-4} du \\ &= -3 \left(\frac{1}{2u^2} - \frac{1}{3u^3} \right) \Big|_1^\infty = 3 \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{2}.\end{aligned}$$

With a similar idea, we have

$$\begin{aligned}E(X^2) &= \int_0^\infty \frac{3t^2}{(1+t)^4} dt = 3 \int_1^\infty \frac{(u-1)^2}{u^4} du \\ &= 3 \int_1^\infty \frac{u^2 - 2u + 1}{u^4} du \\ &= -3 \left(\frac{1}{u} - \frac{1}{u^2} + \frac{1}{3u^3} \right) \Big|_1^\infty = 3 \left(\frac{1}{1} - \frac{1}{1} + \frac{1}{3} \right) = 1\end{aligned}$$

Therefore

$$\sigma^2 = E(X^2) - \mu^2 = 1 - \left(\frac{1}{2} \right)^2 = \frac{3}{4}.$$

Problem 17. (Exercise 10)

Let X and Y be independent random variables with uniform density functions on $[0, 1]$. Find

- (a) $E(|X - Y|)$.
- (b) $E(\max(X, Y))$.
- (c) $E(\min(X, Y))$.
- (d) $E(X^2 + Y^2)$.
- (e) $E((X + Y)^2)$.

Solution 17.

(a) We have

$$E(|X - Y|) = \int_0^1 \int_0^1 |x - y| dy dx.$$

We will use that $|x - y| = x - y$ when $x > y$ and it is equal to $y - x$ when $y > x$. Therefore

$$\begin{aligned}E(|X - Y|) &= \int_0^1 \int_0^x (x - y) dy dx + \int_0^1 \int_x^1 (y - x) dy dx \\ &= \int_0^1 x^2 - \frac{x^2}{2} dx + \int_0^1 \left(\frac{1 - x^2}{2} \right) - x(1 - x) dx \\ &= \int_0^1 x^2 - x + \frac{1}{2} dx = \frac{1}{3}.\end{aligned}$$

(b) We have

$$E(\max(X, Y)) = \int_0^1 \int_0^1 \max(x, y) dy dx.$$

We will use that $\max(x, y) = x$ when $x > y$ and $\max(x, y) = y$ when $y > x$. Therefore

$$\begin{aligned} E(\max(X, Y)) &= \int_0^1 \int_0^x x \, dy \, dx + \int_0^1 \int_x^1 y \, dy \, dx \\ &= \int_0^1 x^2 \, dx + \int_0^1 \left(\frac{1-x^2}{2} \right) \, dx = \int_0^1 \frac{1+x^2}{2} \, dx \\ &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}. \end{aligned}$$

(c) We have $\max(x, y) + \min(x, y) = x + y$, so

$$E(X + Y) = E(\max(X, Y)) + E(\min(X, Y)).$$

But

$$E(X + Y) = E(X) + E(Y) = \frac{1}{2} + \frac{1}{2} = 1.$$

It follows that

$$E(\min(X, Y)) = 1 - E(\max(X, Y)) = 1 - \frac{2}{3} = \frac{1}{3}.$$

(d) We have

$$E(X^2 + Y^2) = \int_0^1 \int_0^1 (x^2 + y^2) \, dy \, dx = \int_0^1 x^2 + \frac{1}{3} \, dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

(e) We have

$$E((X + Y)^2) = E(X^2) + 2E(XY) + E(Y^2).$$

But

$$E(Y^2) = E(X^2) = \int_0^1 x^2 \, dx = \frac{1}{3}.$$

Also, X and Y are independent, so

$$E(XY) = E(X)E(Y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Therefore

$$E((X + Y)^2) = \frac{1}{3} + 2 \cdot \frac{1}{4} + \frac{1}{3} = \frac{7}{6}.$$

Problem 18. (Exercise 26)

(from Hamming) Suppose you are standing on the bank of a straight river.

- Choose, at random, a direction which will keep you on dry land, and walk 1 km in that direction. Let P denote your position. What is the expected distance from P to the river?
- Now suppose you proceed as in part (a), but when you get to P , you pick a random direction (from among all directions) and walk 1 km. What is the probability that you will reach the river before the second walk is completed?

Solution 18.

- Let θ be the angle created with respect to the orthogonal direction to the river. Therefore

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

We can therefore model it with a uniform distribution in the interval

$$\left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

The distance to the river would be $\cos(\theta)$. Therefore, the expected distance is

$$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) \, d\theta = \frac{2}{\pi}.$$

- (b) As in (a), we can model the first step depending on $\theta \in [-\pi/2, \pi/2]$. Once we get to a point P , then to reach the river, the direction has to be in the other direction between $-\theta$ and θ . Therefore, out of 2π possible directions for the second movement 2θ of them lead to the river. Therefore the probability is

$$\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{2|\theta|}{2\pi} d\theta = \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} |\theta| d\theta = \frac{2}{\pi^2} \int_0^{\pi/2} \theta d\theta = \frac{2}{\pi^2} \left(\frac{(\pi/2)^2}{2} \right) = \frac{1}{4}.$$

