Report on the 64th Annual International Mathematical Olympiad

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The International Mathematical Olympiad (IMO) is the world's leading mathematics competition for high school students and is organized annually by different host countries. The competition consists of three problems each on two consecutive days, with an allowed time of four and a half hours both days. In recent years, more than one hundred countries have sent teams of up to six students to compete.

The 64th IMO was organized by Japan, and it was held in Chiba between July 2 and July 13, 2023, with the participation of 618 contestants from 112 countries.

Each year, the members of the US team are chosen during the Math Olympiad Program (MOP), a year-long endeavor organized by the MAA's American Mathematics Competitions (AMC) program. Students gain admittance to MOP based on their performance on a series of examinations, culminating in the USA Mathematical Olympiad (USAMO). A report on the 2023 USAMO can be found in the February 2024 issue of this *Magazine*; a similar report on the 2023 USA Junior Mathematical Olympiad appeared in the January 2024 issue of the *College Mathematics Journal*. More information on the American Mathematics Competitions program can be found on the site `https://www.maa.org/math-competitions`.

The members of the 2023 US team were Jeff Lin (12th grade, Lexington High School, MA); Derek Liu (12th grade, Torrey Pines High School, CA); Maximus Lu (12th grade, Syosset High School, NY); Eric Shen (12th grade, Lynbrook High School, CA); Alexander Wang (9th grade, Millburn High School, NJ); and Alex Zhao (11th grade, Lakeside School, WA). Lin, Liu, Shen, Wang, and Zhao each earned Gold Medals; and Lu received a Silver Medal. In the unofficial ranking of countries, the United States finished second after China.

Below we present the problems and solutions of the 64th IMO. Our solutions are those of the current authors, utilizing some of the various sources already available. Each problem was worth 7 points; the nine-tuple $(a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0; \mathbf{a})$ states the number of students who scored $7, 6, \ldots, 0$ points, respectively, followed by the mean score achieved for the problem.

Problem 1 $(474, 8, 6, 9, 9, 67, 19, 26; \mathbf{5.845})$; *proposed by Columbia.* Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.

First solution. We will prove that the only integers that satisfy the requirements are the second or higher powers of primes. It is easy to see that these numbers work: If $n = p^{k-1}$ for $k \geq 3$ and some prime p, then $d_i = p^{i-1}$ for all $1 \leq i \leq k$, so the condition holds.

Suppose now that n has at least two different prime divisors, and consider its factorization

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$$

with primes $p_1 < p_2 < \cdots < p_m$ and positive exponents $\alpha_1, \alpha_2, \ldots, \alpha_m$. Let j be the largest integer with $1 \leq j \leq \alpha_1$ for which p_1^j is less than p_2. This determines the first and last $j+2$ positive divisors of n, in particular, $d_j = p_1^{j-1}$, $d_{j+1} = p_1^j$, $d_{j+2} = p_2$, and $d_{k-j-1} = n/p_2$, $d_{k-j} = n/p_1^j$, and $d_{k-j+1} = n/p_1^{j-1}$. If n satisfies the given condition, then d_{k-j-1} divides $d_{k-j} + d_{k-j+1}$, and thus n/p_2 divides $n/p_1^j + n/p_1^{j-1}$, which we can rearrange and simplify to write that p_1^j divides $p_2(1+p_1)$. Since this is a contradiction, no value of n with two or more different prime divisors satisfies our requirements.

Second solution. We will prove that the given conditions imply that d_i divides d_{i+1} for every $1 \leq i \leq k-1$. We proceed by induction. Our claim trivially holds for $i = 1$. Let us assume then that d_j divides d_{j+1} for some $1 \leq j \leq k-2$; we need to prove that d_{j+1} divides d_{j+2}.

According to our conditions, d_j divides $d_{j+1} + d_{j+2}$, and since we are assuming that d_j divides d_{j+1}, we get that d_j divides d_{j+2}. Note that the positive divisors of n come in pairs; that is, we have $d_i \cdot d_{k-i+1} = n$ for each $i = 1, 2, \ldots, k$. (This observation holds even in the case where n is a square number where the divisor $d_{(k+1)/2}$ is its own pair.) Therefore, we get that n/d_{k-j+1} divides n/d_{k-j-1}, and so d_{k-j-1} divides d_{k-j+1}.

Returning to the given conditions again, we have that d_{k-j-1} divides $d_{k-j} + d_{k-j+1}$. Therefore, d_{k-j-1} divides d_{k-j}, that is, n/d_{j+1} divides n/d_{j+2}. But this implies that d_{j+1} divides d_{j+2}, as claimed.

We thus see that d_{i_1} divides d_{i_2} for every $1 \leq i_1 \leq i_2 \leq k$. But then n cannot have two distinct prime divisors since neither can divide the other, and thus $n = p^{k-1}$ for some $k \geq 3$ and prime p.

Problem 2 $(215, 6, 7, 20, 62, 6, 100, 202; \mathbf{3.162})$; *proposed by Portugal.* Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Solution. Let M be the antipode of S on Ω. Then arcs BM and MC are equal, and thus AM is the angle bisector of $\angle BAC$. Define $X = \overline{AM} \cap \overline{BS}$; our goal is to prove that PX is tangent to ω at P.

We first show that L, P, and S are collinear. Indeed, we have

$$\angle LPB = \angle LDB = \angle CBD = \angle CBS = \angle SCB = 180^\circ - \angle SPB,$$

which implies our claim.

In a similar manner, we let F be the antipode of A on Ω, so that quadrilateral $AMFS$ is in fact a rectangle. We also have $\angle AEF = 90^\circ$, meaning both $\overline{EF}$ and $\overline{BC}$ are perpendicular to $\overline{AE}$, in particular, they are parallel and hence arcs ME and MF on Ω are equal. Therefore,

$$\angle SPD = 180^\circ - \angle DPL = \angle LBD = 180^\circ - \angle EBS = 180^\circ - \angle SCF = \angle FPS,$$

and hence P, D, and F are collinear.

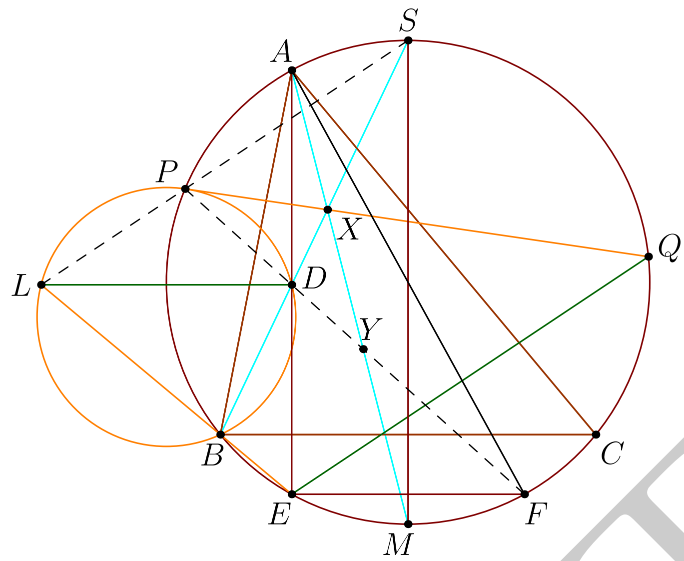

Figure 1: Illustration for Problem 2.

Since AE is parallel to SM, we find that $\angle EAM = \angle AMS$ and hence triangles AXD and MXS are similar; setting $Y = \overline{PF} \cap \overline{AM}$, we see that triangles XDY and SDF are similar as well. From this we get

$$\frac{AX}{XD} = \frac{AX + XM}{XD + SX} = \frac{AM}{SD} = \frac{SF}{SD} = \frac{XY}{XD}.$$

Therefore, $AX = XY$. Since AF is a diameter, we have $\angle APY = \angle APF = 90^\circ$, and we conclude that $XP = XA$.

Now extend ray PX to meet Ω again at Q. From $XP = XA$, we conclude that arcs PM and AQ are equal. Since arcs AS and EM are also equal, it follows that arcs PE and SQ are equal as well. Then

$$180^\circ - \angle LPQ = \angle QPS = \angle PQE = \angle PFE = \angle PDL,$$

yielding the desired tangency.

Problem 3 $(73, 3, 6, 8, 23, 7, 102, 396; \mathbf{1.256})$; *proposed by Malaysia.* For each integer $k \geqslant 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2} \cdots a_{n+k}$$

for every integer $n \geqslant 1$.

Solution. We claim that the sequences satisfying the requirement are the arithmetic progressions with positive integer terms. (While the problem statement excludes $k = 1$, this claim holds then as well, providing perhaps a hint for the general answer.) It is easy to see that these work, since if there is some nonnegative integer d so that $a_n = a_1 + (n-1)d$ for all positive integers n, then one can take $P(x) = (x+d)(x+2d)\cdots(x+kd)$. In particular, constant sequences with positive integer values satisfy the requirements with $P(x) = x^k$, so below we assume that $\mathbf{a} = (a_1, a_2, \ldots)$ is an infinite non-constant sequence of positive integers for which the requirements hold with a polynomial P, and prove that $\mathbf{a}$ is indeed an arithmetic progression.

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We start by proving that $\mathbf{a}$ is increasing, that is, $a_n \le a_{n+1}$ for every positive integer n. For that purpose, we let m_n denote for a given positive integer n the minimum value of the set $\{a_i \mid i \ge n\}$; our goal is to show that $a_n = m_n$ for every n. If this were not the case, then we could find an index $j_n > n$ so that $a_{j_n} = m_n$ but $a_i > m_n$ for each $n \le i < j_n$. In particular, we would have $a_{j_n - 1} > a_{j_n}$, and since P is a strictly increasing function, this would imply that $P(a_{j_n-1}) > P(a_{j_n})$. This means that

$$a_{j_n} a_{j_n+1} \cdots a_{j_n+k-1} > a_{j_n+1} a_{j_n+2} \cdots a_{j_n+k},$$

implying that $a_{j_n} > a_{j_n+k}$, a contradiction. Therefore, $a_n = \min\{a_i \mid i \ge n\}$ for every positive integer n, and thus $\mathbf{a}$ is increasing, as claimed.

Next, we examine how fast $\mathbf{a}$ is increasing. Let C be an integer for which $\binom{k}{i} C^i$ is at least c_{k-i} for every $i = 1, 2, \ldots, k$. We claim that $a_n + 1 \le a_{n+1} \le a_n + C$ for every n. The upper bound is easy to see: since our sequence is increasing, we have

$$a_{n+1}^k \le a_{n+1} a_{n+2} \cdots a_{n+k} = P(a_n) = a_n^k + c_{k-1} a_n^{k-1} + \cdots + c_1 a_n + c_0 \le (a_n + C)^k,$$

so $a_{n+1} \le a_n + C$.

To prove the lower bound, we show that if $a_{n_0} = a_{n_0+1}$ were to hold for some positive integer n_0, then $a_n = a_{n+1}$ for all n, contradicting our assumption that $\mathbf{a}$ is a non-constant sequence. Indeed, if $a_{n_0} = a_{n_0+1}$, then $P(a_{n_0}) = P(a_{n_0+1})$, so

$$a_{n_0+1} a_{n_0+2} \cdots a_{n_0+k} = a_{n_0+2} a_{n_0+3} \cdots a_{n_0+k+1},$$

and thus $a_{n_0+1} = a_{n_0+k+1}$. But $\mathbf{a}$ is increasing, so this can only happen if $a_i = a_{n_0}$ for all $n_0 \le i \le n_0 + k + 1$. But then $P(a_{n_0}) = a_{n_0}^k$, and since $c_0, \ldots, c_{k-1}$ are nonnegative, this implies that $P(x) = x^k$ for all x. Taking now an arbitrary positive integer n, we have $P(a_n) = a_n^k$, and so

$$a_{n+1} a_{n+2} \cdots a_{n+k} = a_n^k,$$

which can only happen if $a_i = a_n$ for all $n \le i \le n + k + 1$, in particular if $a_n = a_{n+1}$, as claimed.

Setting

$$U = [1, C] \times [1, 2C] \times \cdots \times [1, kC] \times [1, (k+1)C],$$

we thus get that

$$\mathbf{\Delta_n} = (a_{n+1} - a_n, a_{n+2} - a_n, \ldots, a_{n+k} - a_n, a_{n+k+1} - a_n) \in U$$

for every positive integer n. But U contains only finite many points with integer coordinates, so there must exist an element $\mathbf{\Delta} \in U$ for which the set $M = \{m \in \mathbb{N} \mid \mathbf{\Delta_m} = \mathbf{\Delta}\}$ is infinite. With $\mathbf{\Delta} = (\delta_1, \delta_2, \ldots, \delta_k, \delta_{k+1})$, for each $m \in M$ we can thus write

$$P(a_m) = (a_m + \delta_1)(a_m + \delta_2) \cdots (a_m + \delta_k)$$

and

$$P(a_m + \delta_1) = (a_m + \delta_2)(a_m + \delta_3) \cdots (a_m + \delta_{k+1}).$$

Note that a polynomial of degree k may have at most k roots. Therefore, the fact that both of these equations hold for infinitely many different m and thus (since $\mathbf{a}$ is strictly increasing) for infinitely many different values of a_m, means that

$$P(x) = (x + \delta_1)(x + \delta_2) \cdots (x + \delta_k)$$

and

$$P(y + \delta_1) = (y + \delta_2)(y + \delta_3) \cdots (y + \delta_{k+1})$$

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for every x and y, thus with $y = x - \delta_1$ we get the polynomial equation

$$(x+\delta_1)(x+\delta_2)\cdots(x+\delta_k) = (x+\delta_2-\delta_1)(x+\delta_3-\delta_1)\cdots(x+\delta_{k+1}-\delta_1).$$

Since $\delta_1 < \delta_2 < \cdots < \delta_k$, this implies that $\delta_i = \delta_{i+1} - \delta_1$ for each $1 \le i \le k$, and therefore $\delta_i = i \cdot \delta_1$ for each $1 \le i \le k+1$.

This means that for each $m \in M$, $(a_m, a_{m+1}, \ldots, a_{m+k+1})$ is an arithmetic progression with the same common difference d (where $d = \delta_1$). To complete our proof, we will need to establish that M contains every positive integer. Since M is an infinite set, we only need to verify that for every $m \ge 2$, if $m \in M$ then $m-1 \in M$. To do so, we evaluate $P(a_{m-1})$ in two different ways, first by using the condition given in the problem and that $m \in M$, and then by our expression for $P(x)$ for $x = a_{m-1}$. This yields

$$a_m(a_m+d)\cdots(a_m+(k-1)d) = (a_{m-1}+d)(a_{m-1}+2d)\cdots(a_{m-1}+kd).$$

It is now easy to see that we must have $a_m = a_{m-1}+d$, and thus $(a_{m-1}, a_m, \ldots, a_{m+k})$ is an arithmetic progression with common difference d. This completes our proof.

Problem 4 $(384, 3, 1, 4, 8, 32, 100, 86; \mathbf{4.717})$; *proposed by the Netherlands.* Let $x_1, x_2, \cdots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \cdots, 2023$. Prove that $a_{2023} \ge 3034$.

Solution. We will prove that for every positive integer $k \le 1012$, we have $a_{2k-1} \ge 3k-2$ (and thus $a_{2023} \ge 3034$). Since $a_1 = 1$, our inequality holds for $k = 1$; let us assume then that it holds for some $k \le 1011$, and attempt to prove that $a_{2k+1} \ge 3k+1$.

Using the notations $A = x_1 + x_2 + \cdots + x_{2k-1}$ and $B = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{2k-1}}$, we have

$$\begin{aligned} a_{2k+1}^2 &= (A + x_{2k} + x_{2k+1})\left(B + \frac{1}{x_{2k}} + \frac{1}{x_{2k+1}}\right) \\ &= a_{2k-1}^2 + x_{2k}B + \frac{1}{x_{2k}}A + x_{2k+1}B + \frac{1}{x_{2k+1}}A + \frac{x_{2k}}{x_{2k+1}} + \frac{x_{2k+1}}{x_{2k}} + 2. \end{aligned}$$

By the AM-GM inequality, we see that

$$x_{2k}B + \frac{1}{x_{2k}}A \ge 2\sqrt{AB} = 2a_{2k-1},$$

$$x_{2k+1}B + \frac{1}{x_{2k+1}}A \ge 2\sqrt{AB} = 2a_{2k-1},$$

and

$$\frac{x_{2k}}{x_{2k+1}} + \frac{x_{2k+1}}{x_{2k}} > 2$$

(note that our last inequality is strict as $x_{2k} \neq x_{2k+1}$). But then

$$a_{2k+1}^2 > a_{2k-1}^2 + 4a_{2k-1} + 4 = (a_{2k-1}+2)^2.$$

Recalling now that a_n is an integer for all n, we can write $a_{2k+1} \ge a_{2k-1} + 3$, so by our inductive assumption we conclude that $a_{2k+1} \ge 3k+1$, as claimed.

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Problem 5 $(118, 9, 13, 4, 52, 174, 29, 219; \mathbf{2.417})$; *proposed by the Netherlands.* Let n be a positive integer. A Japanese triangle consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.

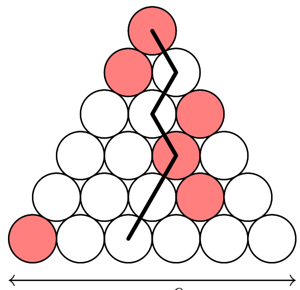

Figure 2: Illustration for Problem 5.

In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k circles.

First solution. The answer is $k = \lfloor \log_2(n) \rfloor + 1$.

To show that this value of k is achievable, we construct a Japanese triangle that does not have a ninja path containing more than $\lfloor \log_2(n) \rfloor + 1$ red circles. Let $m = \lfloor \log_2(n) \rfloor$; we then have $2^m \leq n < 2^{m+1}$. We label the circles in the triangle as (i, j) if they are in the ith row (from top to bottom) and jth column (from left to right). Let $0 \leq a \leq m - 1$, then for row i satisfying $2^a \leq i < 2^{a+1}$, we color (i, j) red where $j = 2(i - 2^a) + 1$. Similarly, for $2^m \leq i \leq n$, we color (i, j) red where $j = 2(i - 2^m) + 1$. Note that the coloring is possible because $i \leq 2^{a+1} - 1$ implies that $j = 2(i - 2^a) + 1 \leq i$. (See Figure 3 for an example.) By construction, each ninja path can cover at most one red circle from row 2^a to row $2^{a+1} - 1$ for each $0 \leq a \leq m$, because each ball is 2 away from the previous one in the same direction. Therefore, $k \leq m + 1 = \lfloor \log_2(n) \rfloor + 1$.

Now we need to prove that, regardless of the coloring of the circles, there is a ninja path containing at least $\lfloor \log_2(n) \rfloor + 1$ red circles. For that purpose, recall that the *height* of a finite partially ordered set (poset), defined as the length of its longest chain, is equal to the minimum number of antichains that can partition the poset.[1]

Let P be the poset consisting of the red circles of our Japanese triangle, with a red circle at (i_1, j_1) defined as *above* a red circle at (i_2, j_2) if $i_1 \leq i_2$ and there is a ninja path containing them. It is easy to see that P is reflexive, antisymmetric and transitive, so it is a poset. Our goal is to prove that if P has height h, then it has size at most $2^h - 1$. This will imply our claim, since P has n elements, and so its height is at least $\lfloor \log_2(n) \rfloor + 1$.

We proceed by induction on h. For $h = 1$, because the circle $(1, 1)$ has to be colored red and it is above every other red circle, the poset consists of a single element and thus has size $2^1 - 1 = 1$. Suppose now that the statement is true for a positive integer h, and consider P that has height $h + 1$, that is, the minimum number of antichains that can partition P is $h + 1$. We label these antichains $A_1, A_2, \ldots, A_{h+1}$ so that the smallest row index in A_1

[1] This statement easily follows from the fact that the maximal elements of the poset form an antichain. The dual statement, that the size of the largest antichain equals the minimum number of chains that can partition the poset, is known as Dilworth's Theorem, and is considerably harder to establish.

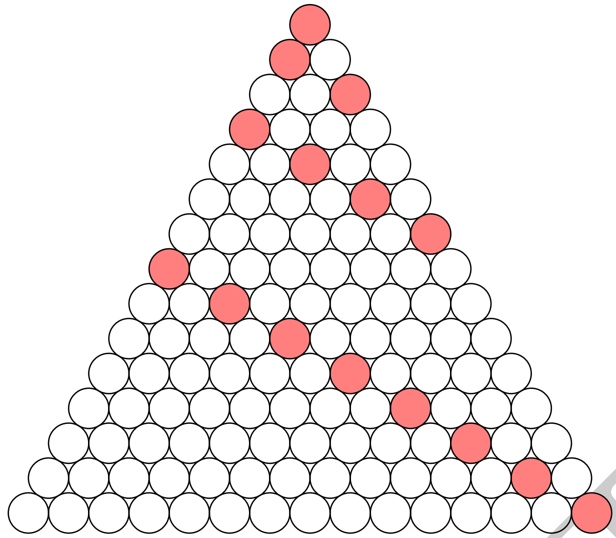

Figure 3: An example for a Japanese triangle with $n = 15$ where no ninja path contains more than 4 red circles.

is less than or equal to the smallest row index in A_2, which is less than or equal to the smallest row index in A_3, and so on. By our induction hypothesis, $A_1 \cup A_2 \cup \cdots \cup A_h$ has at most $2^h - 1$ elements.

Suppose that the red circle in A_{h+1} with smallest row index is labeled (r, c); we then have $r \leq 2^h$. Consider now the following three sets of circles from our Japanese triangle:

$$\begin{aligned} C_1 &= \{(i,j) \mid i \geq r, j < c\} \\ C_2 &= \{(i,j) \mid i \geq r, c \leq j \leq c + i - r\} \\ C_3 &= \{(i,j) \mid i \geq r, j > c + i - r\}. \end{aligned}$$

Note that the only element of C_2 that is in A_{h+1} is (r, c). Furthermore, C_1 is the union of $c - 1$ chains and C_3 is the union of $r - c$ chains, and thus A_{h+1} can contain at most $c - 1$ elements of C_1 and at most $r - c$ elements of C_3. Therefore, $P = A_1 \cup A_2 \cup \cdots \cup A_{h+1}$ has at most

$$(2^h - 1) + (c - 1) + 1 + (r - c) = 2^h - 1 + r \leq 2^h - 1 + 2^h = 2^{h+1} - 1$$

elements, as needed. This completes our proof.

Second solution. We construct a rooted binary tree T_1 on the set of all circles as follows. For each row other than the bottom row:

- Connect the red circle to both circles under it.
- White circles to the left of the red circle in their row are connected to the left.
- White circles to the right of the red circle in their row are connected to the right.

The circles in the bottom row are all leaves of this tree. (See Figure 4 for an example with $n = 6$.) Note that this construction indeed results in a tree since no circle can be connected to both circles directly above it.

Next, we construct a new tree from T_1, labeled T_2, that contains only the red circles, with two red circles connected if there is a path of (zero or more) white circles connecting them in T_1. (Here T_2 is a graph minor of T_1. See example in Figure 4.) Our goal is to

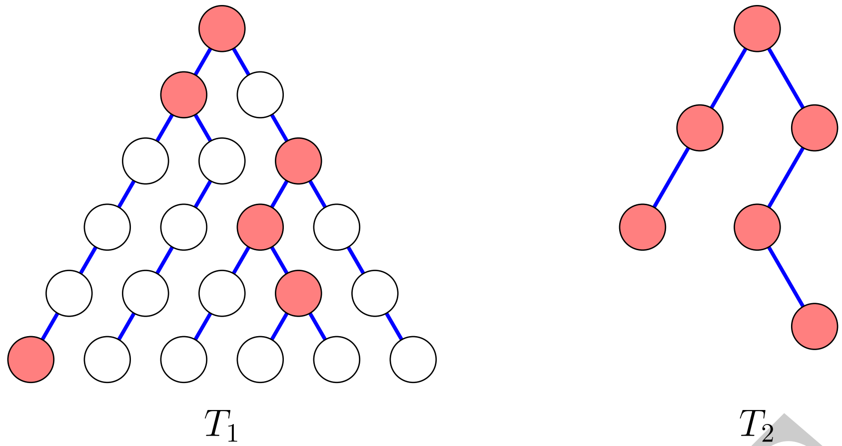

Figure 4: An illustration for the second solution to Problem 5.

establish that T_2 has a chain of length at least $\lfloor \log_2(n) \rfloor + 1$. This follows easily, since the number of nodes in a rooted binary tree of depth d can be at most $1 + 2 + 4 + \cdots + 2^{d-1}$, so we must have $n \le 2^d - 1$. As our construction in the first solution demonstrates, there exist Japanese triangles where no ninja path contains more than $\lfloor \log_2(n) \rfloor + 1$ red circles, so our proof is complete.

Problem 6 $(6, 4, 1, 1, 4, 36, 11, 555; \mathbf{0.275})$; *proposed by the United States of America.* Let ABC be an equilateral triangle.

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let $A_2 = \overline{BC_1} \cap \overline{CB_1}$, $B_2 = \overline{CA_1} \cap \overline{AC_1}$, $C_2 = \overline{AB_1} \cap \overline{BA_1}$. Prove that if triangle $A_1B_1C_1$ is scalene, then the circumcircles of triangles AA_1A_2, BB_1B_2, and CC_1C_2 all pass through two common points.

Solution. Define O as the circumcenter of $\triangle ABC$ and set the angles

$$\begin{aligned}
\alpha &= \angle A_1CB = \angle CBA_1, \\
\beta &= \angle ACB_1 = \angle B_1AC, \\
\gamma &= \angle C_1AB = \angle C_1BA.
\end{aligned}$$

Then

$$\alpha + \beta + \gamma = 30^\circ.$$

In particular, $\max(\alpha, \beta, \gamma) < 30^\circ$, so it follows that A_1 lies inside $\triangle OBC$, and similarly for the others. This means, for example, that C_1 lies between B and A_2, and so on. Therefore the polygon $A_2C_1B_2A_1C_2B_1$ is convex.

Note that $\angle BA_1C = 180^\circ - 2\alpha$, and

$$\begin{aligned}
\angle BA_2C &= 180^\circ - \angle C_1BC - \angle B_1CB \\
&= 180^\circ - (60^\circ - \gamma) - (60^\circ - \beta) \\
&= 60^\circ + \beta + \gamma = 90^\circ - \alpha = \frac{1}{2}\angle BA_1C.
\end{aligned}$$

Since A_1 lies inside $\triangle BA_2C$, it follows that A_1 is exactly the circumcenter of $\triangle A_2BC$. Analogously, we can also deduce that B_1 is the circumcenter of $\triangle B_2CA$, and C_1 is the circumcenter of $\triangle C_2AB$.

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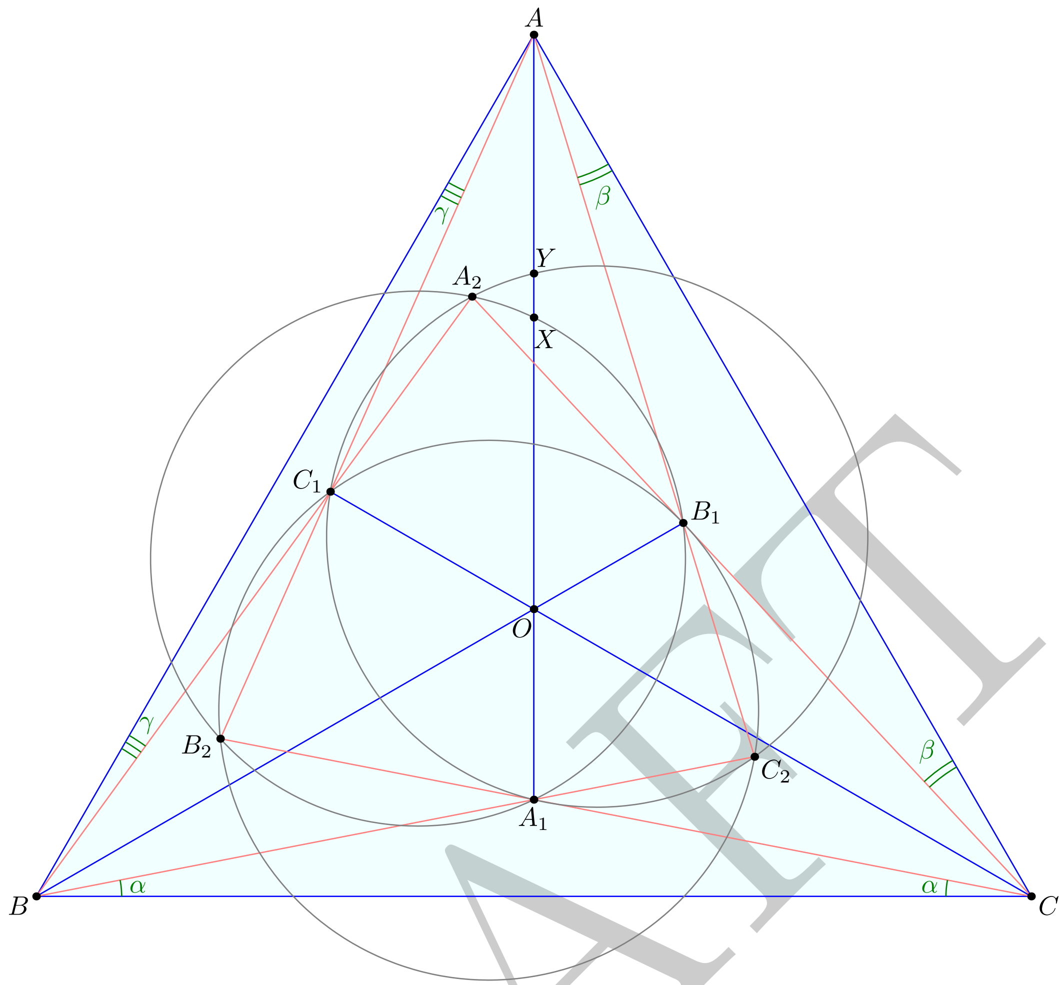

Figure 5: An illustration for the solution to Problem 6.

Since A_1 is the circumcenter of $\triangle A_2BC$, $\angle CA_1A_2 = 2\angle CBA_2$. Therefore

$$\angle A_2A_1B_2 = 180^\circ - \angle CA_1A_2 = 180^\circ - 2\angle CBA_2 = 180^\circ - 2(60^\circ - \gamma) = 60^\circ + 2\gamma.$$

Similarly, using that B_1 is the circumcenter of $\triangle B_2AC$,

$$\angle A_2B_1B_2 = 180^\circ - \angle CB_1B_2 = 180^\circ - 2\angle CAB_2 = 180^\circ - 2(60^\circ - \gamma) = 60^\circ + 2\gamma.$$

Since $\angle A_2A_1B_2 = \angle A_2B_1B_2$, we have that $A_2B_1A_1B_2$ is cyclic. Analogously, $B_2C_1B_1C_2$ and $A_1C_1A_2C_2$ are cyclic.

Let γ_a be the circle circumscribing quadrilateral $B_2C_1B_1C_2$, γ_b the circle for $A_1C_1A_2C_2$, and γ_c the circle for $A_2B_1A_1B_2$. We will show that these three circles are distinct. Assume, for the sake of contradiction, that the convex hexagon $A_2C_1B_2A_1C_2B_1$ is cyclic. Then

$$360^\circ = \angle C_2A_1B_1 + \angle B_2C_1A_2 + \angle A_2B_1C_2 = \angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ,$$

which is absurd. This contradiction eliminates the degenerate case, so the three circles are distinct.

For the remainder of the solution, let $\mathrm{Pow}(P, \omega)$ denote the power of a point P with respect to a circle ω. In other words, if O_ω is the center of circle ω, and r_ω is the radius of circle ω, then $\mathrm{Pow}(P, \omega)$ is defined as $|PO_\omega|^2 - r_\omega^2$. An important property of the power of a point is that if P is outside of ω and a ray from P to ω intersects ω at S and T, then $PS \cdot PT = \operatorname{Pow}(P, \omega)$.

Let line AA_1 meet γ_b and γ_c again at X and Y, and set $k_a = \frac{AX}{AY}$. Consider the set C_a of all points P satisfying $\operatorname{Pow}(P, \gamma_b) = k_a \operatorname{Pow}(P, \gamma_c)$. We now invoke a fact known as the Coaxiality Lemma[2], which states that (given γ_b and γ_c are not concentric) the locus $\mathcal{C}_a$ is

- a circle if $k_a \neq 1$; and
- a line if $k_a = 1$ (in which case it is the radical axis).

Consider the points A_1, A_2, and A. Since A_1, A_2 are in γ_b, γ_c, we have $\operatorname{Pow}(A_1, \gamma_b) = 0 = k_a \operatorname{Pow}(A_1, \gamma_c)$ and $\operatorname{Pow}(A_2, \gamma_b) = 0 = k_a \operatorname{Pow}(A_2, \gamma_c)$. Therefore $A_1, A_2 \in \mathcal{C}_a$. For the point A, we have

$$\operatorname{Pow}(A, \gamma_b) = AX \cdot AA_1 = \frac{AX}{AY} \cdot AY \cdot AA_1 = k_a \cdot AY \cdot AA_1 = k_a \operatorname{Pow}(A, \gamma_c).$$

Therefore $A \in \mathcal{C}_a$. Hence A, A_1, and A_2 are in $\mathcal{C}_a$. Since these three points are not collinear, $k_a \neq 1$, and $\mathcal{C}_a$ must be exactly the circumcircle of $\triangle AA_1A_2$ from the problem statement.

We turn to evaluating k_a more carefully. Using that B_1 is the circumcenter of $\triangle B_2AC$, we have $\angle CB_2B_1 = 90^\circ - \angle B_2AC$. Therefore,

$$\angle A_1XB_1 = \angle A_1B_2B_1 = \angle CB_2B_1 = 90^\circ - \angle B_2AC = 90^\circ - (60^\circ - \gamma) = 30^\circ + \gamma.$$

Now using the law of sines, using that $\angle A_1XB_1 + \angle AXB_1 = 180^\circ$, and several angles that we have calculated already,

$$\begin{aligned}\frac{AX}{AB_1} &= \frac{\sin \angle AB_1X}{\sin \angle AXB_1} = \frac{\sin(\angle A_1XB_1 - \angle XAB_1)}{\sin \angle AXB_1} = \frac{\sin(\angle A_1XB_1 - \angle XAB_1)}{\sin \angle A_1XB_1} \\ &= \frac{\sin\left((30^\circ + \gamma) - (30^\circ - \beta)\right)}{\sin(30^\circ + \gamma)} = \frac{\sin(\beta + \gamma)}{\sin(30^\circ + \gamma)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{AY}{AC_1} &= \frac{\sin \angle AC_1Y}{\sin \angle AYC_1} = \frac{\sin(\angle A_1YC_1 - \angle YAC_1)}{\sin \angle AYC_1} = \frac{\sin(\angle A_1YC_1 - \angle YAC_1)}{\sin \angle A_1CB_1} \\ &= \frac{\sin\left((30^\circ + \beta) - (30^\circ - \gamma)\right)}{\sin(30^\circ + \beta)} = \frac{\sin(\beta + \gamma)}{\sin(30^\circ + \beta)}.\end{aligned}$$

Therefore,

$$k_a = \frac{AX}{AY} = \frac{AB_1}{AC_1} \cdot \frac{\sin(30^\circ + \beta)}{\sin(30^\circ + \gamma)}.$$

Now define analogous constants k_b and k_c and circles $\mathcal{C}_b$ and $\mathcal{C}_c$. Owing to the symmetry of the expressions, we have the key relation $k_ak_bk_c = 1$.

In summary, the three circles in the problem statement may be described as

$$\begin{aligned}\mathcal{C}_a &= (AA_1A_2) = \{\text{points } P \text{ such that } \operatorname{Pow}(P, \gamma_b) = k_a \operatorname{Pow}(P, \gamma_c)\}, \\ \mathcal{C}_b &= (BB_1B_2) = \{\text{points } P \text{ such that } \operatorname{Pow}(P, \gamma_c) = k_b \operatorname{Pow}(P, \gamma_a)\}, \\ \mathcal{C}_c &= (CC_1C_2) = \{\text{points } P \text{ such that } \operatorname{Pow}(P, \gamma_a) = k_c \operatorname{Pow}(P, \gamma_b)\}.\end{aligned}$$

Since k_a, k_b, and k_c have product 1, it follows that any point on at least two of the circles must lie on the third circle as well. The convexity of hexagon $A_2C_1B_2A_1C_2B_1$ mentioned

[2]The proof can be readily established using the Cartesian coordinate system by noticing that for a point $P = (x, y)$ and a circle ω, $Pow(P, \omega) = x^2 + y^2 + Ax + By + C$, where A, B, and C depend on the center and radius of ω.

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earlier ensures that any two of these circles intersect at two different points, completing the proof.

Summary. We present the problems and solutions to the 64th Annual International Mathematical Olympiad.

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