Report on the 16th Annual USA Junior Mathematical Olympiad

Béla Bajnok, John Berman, and Enrique Treviño

Photo

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The USA Junior Mathematical Olympiad (USAJMO) is the final round in the American Mathematics Competitions series for high school students in grade 10 or below, organized each year by the Mathematical Association of America. The competition follows the style of the International Mathematical Olympiad (IMO): it consists of three problems each on two consecutive days, with an allowed time of four and a half hours both days.

The 16th annual USAJMO was given on Tuesday, March 19 and Wednesday, March 20, 2025, and was taken by 241 students. Further information on the American Mathematics Competitions program can be found on the site https://maa.org/student-programs/amc/. Below we present the problems and solutions of the competition; a similar article for the USA Mathematical Olympiad (USAMO), offered to high school students in grade 12 or below, can be found in a concurrent issue of *Mathematics Magazine*.

The problems of the USAJMO are chosen—from a large collection of proposals submitted for this purpose—by the USAMO/USAJMO Editorial Board that works under the leadership of co-editors-in-chief Oleksandr Rudenko and Enrique Treviño. This year's problems were created by John Berman, Wilbert Chu, Carl Schildkraut, Enrique Treviño, and Hung-Hsun Yu.

The solutions presented here are those of the present authors, relying in part on the submissions of the problem authors. Each problem was worth 7 points; the nine-tuple

$$(n; a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$$

states the number of students who submitted a paper for the relevant problem, followed by the numbers of students who scored $0, 1, \ldots, 7$ points, respectively.

Problem 1 $(227; 102, 26, 9, 1, 1, 9, 2, 77)$; *proposed by John Berman.* Let $\mathbb{Z}$ be the set of integers, and let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a function. Prove that there are infinitely many integers c such that the function $g\colon \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = f(x) + cx$ is not bijective.

Note: A function $g\colon \mathbb{Z} \to \mathbb{Z}$ is *bijective* if for every integer b, there exists exactly one integer a such that $g(a) = b$.

Solution. Let $\mathcal{C}$ be the set of integers c for which the function $f(x) + cx$ is not bijective. Our goal is to prove that $\mathcal{C}$ is an infinite set; for the sake of a contradiction, we assume that $\mathcal{C}$ is finite.

We set

$$\mathcal{D} = \{f(a) - f(a+1) \mid a \in \mathbb{Z}\},$$

and prove that $\mathcal{D} \subseteq \mathcal{C}$. Indeed, if $d = f(a) - f(a+1)$ for an integer a, then adding $f(a+1) + da$ to both sides, we get

$$f(a+1) + d(a+1) = f(a) + da,$$

hence the function $f(x) + dx$ is not injective, and thus $d \in \mathcal{C}$. In particular, we see that $\mathcal{C}$ is not empty, and thus it has a maximum element that we denote by M.

Let us consider now the function $g(x) = f(x) + (M+2)x$. We can show that $g(x)$ is increasing: indeed, since no element of $\mathcal{D}$ is more than M, we have $f(a) - f(a+1) \leq M$ for every integer a, so we find that

$$g(a+1) = f(a+1) + (M+2)(a+1) \geq f(a) + (M+2)a + 2 = g(a) + 2.$$

But this shows not only that $g(x)$ is increasing, but that the consecutive values in its range differ by at last 2. Therefore, $g(x)$ is not surjective, making $M+2$ an element of $\mathcal{C}$, contradicting our indirect assumption.

Problem 2 $(197; 70, 30, 9, 6, 2, 4, 17, 59)$; *proposed by John Berman.* Let k and d be positive integers. Prove that there exists a positive integer N such that for every odd integer $n > N$, the digits in the base-$2n$ representation of n^k are all greater than d.

Solution. We will prove that $N = 2^{k-1}(d+1)$ satisfies the requirement.

Let us first exhibit the base-$2n$ representation of n^k. Let $i = 0, 1, \ldots, k$. We start by letting q_i and r_i be the quotient and the remainder of n^k when divided by $(2n)^i$; to be more explicit, we can write $n^k = q_i(2n)^i + r_i$ where $q_i = \lfloor n^k/(2n)^i \rfloor$ and r_i is an integer with $0 \leq r_i < (2n)^i$. Observe that the sequence $r_0, \ldots, r_k$ is non-decreasing with $r_0 = 0$ and $r_k = n^k$. In terms of these notations, we can express the i-th digit (with $i = 0, 1, \ldots, k-1$ counting from the right) in the base-$2n$ representation of n^k as

$$d_i = \frac{r_{i+1} - r_i}{(2n)^i}.$$

To verify, note that d_n is an integer since $r_{i+1} - r_i$ is divisible by $(2n)^i$, $0 \le d_i < 2n$, and

$$\sum_{i=0}^{k-1} d_i(2n)^i = \sum_{i=0}^{k-1}(r_{i+1} - r_i) = r_k - r_0 = n^k.$$

Our next goal is to find a lower bound for d_i. For that purpose, we compute

$$n^k - r_i - d_i(2n)^i = (n^k - r_{i+1}) + (r_{i+1} - r_i - d_i(2n)^i) = q_{i+1}(2n)^{i+1}.$$

Since the right side and n^k are both divisible by n^{i+1}, so is $r_i + d_i(2n)^i$. But this quantity is then at least n^{i+1} and less than $(2n)^i + d_i(2n)^i$, which yields $d_i > n/2^i - 1$. Therefore, if $n \ge 2^{k-1}(d+1)$, then all digits in the base-$2n$ representation of n^k are greater than d.

Problem 3 $(175; 143, 3, 12, 2, 1, 3, 1, 10)$; *proposed by Wilbert Chu.* Let m and n be positive integers, and let $\mathcal{R}$ be a $2m \times 2n$ grid of unit squares.

A *domino* is a 1×2 or 2×1 rectangle. A subset S of grid squares in $\mathcal{R}$ is *domino-tileable* if dominoes can be placed to cover every square of S exactly once with no domino extending outside S. *Note*: The empty set is domino-tileable.

An *up-right path* is a path from the lower-left corner of $\mathcal{R}$ to the upper-right corner of $\mathcal{R}$ formed by exactly $2m + 2n$ edges of the grid squares.

Determine, with proof, in terms of m and n, the number of up-right paths that divide $\mathcal{R}$ into two domino-tileable subsets.

Solution. The answer is $\binom{m+n}{m}^2$.

We color the squares of the grid alternatingly white or black (as on a chessboard). If an up-right path divides $\mathcal{R}$ into domino-tileable subsets, then both subsets must have the same number of white squares as black squares. Let us call subsets with the same number of white squares as black squares *balanced*. We will prove that if an up-right path divides $\mathcal{R}$ into two balanced subsets, then both subsets are domino-tileable. Since the total number of black squares in the grid equals the total number of white squares, it suffices to establish that the region below the path is balanced.

We proceed indirectly, and assume that there are up-right paths that divide the grid into two balanced subsets so that the regions below them are not domino-tileable; let P be such an up-right path for which the area below it has minimal area. If at some point in P we have $\uparrow\rightarrow\rightarrow$ (meaning a move up followed by two moves to the right), we can replace it with $\rightarrow\rightarrow\uparrow$ to create a path P' that also divides $\mathcal{R}$ into balanced sets. By our assumption of minimality, the region below P' is domino-tileable, but then the region below P must also be domino-tileable because we can add a horizontal domino in the area created by the change from $\uparrow\rightarrow\rightarrow$ to $\rightarrow\rightarrow\uparrow$. Similarly, if at some point P has $\uparrow\uparrow\rightarrow$, creating P' by changing that part of P with $\rightarrow\uparrow\uparrow$, the region below P' will be balanced and have area less than the area below P, so it will be domino-tileable. Then, the region below P will be domino-tileable by inserting a vertical domino in the region vacated by the change from $\uparrow\uparrow\rightarrow$ to $\rightarrow\uparrow\uparrow$. Therefore, P does not contain steps $\uparrow\rightarrow\rightarrow$ or $\uparrow\uparrow\rightarrow$ consecutively. Observe that if two or more up arrows are not followed by a right arrow, then they must be on the right border, and hence go all the way to the top right corner. Similarly, if two or more right arrows are not preceded by an up arrow, then they must be at the bottom, starting at the bottom left corner. So the only way P does not include either $\uparrow\uparrow\rightarrow$ or $\uparrow\rightarrow\rightarrow$ is if it starts with some (perhaps zero) right arrows, continues with a (perhaps length zero) sequence of moves alternating between

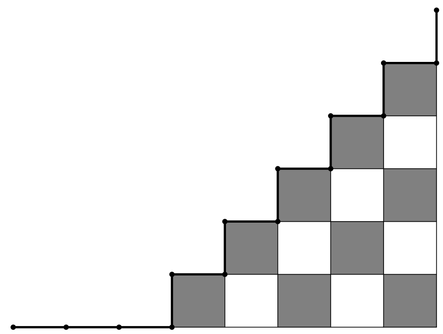

Figure 1: The region below the path →→→↑→↑→↑→↑→↑→↑.

up arrows and right arrows, and then ends with some (perhaps zero) up arrows—see Figure 1 for an illustration.

The initial right arrows and the final up arrows do not add any squares to our region, so the region formed by alternating up arrows and right arrows must be balanced. This is only possible if that region is empty. Indeed, if this were not the case, then the region below P, whose columns share the same color in their top squares, would have more squares of that color. Therefore P consists of $2n$ right arrows followed by $2m$ up arrows, making the region below P empty and thus trivially domino-tileable.

The problem now reduces to counting the number of up-right paths that divide $\mathcal{R}$ into two balanced subsets. We consider the sequence of $2m + 2n$ moves in such a path as a sequence of $m + n$ pairs of *double moves*. There are four types of double moves, namely ↑↑, ↑→, →↑, and →→. We first prove that a path divides $\mathcal{R}$ into two balanced subsets if and only if the number of ↑→ double moves equals the number of →↑ double moves.

Note that the effect of removing a →→ double move from the path, and lowering n accordingly, is to remove two adjacent columns from $\mathcal{R}$. The squares below the path that are removed form a rectangle in which there are an equal number of black and white squares, so such a removal does not change whether the path is balanced. Similarly, if we remove an ↑↑ double move and lower m, we remove two adjacent rows and do not change whether the path is balanced. Thus iterating these removals, we may reduce to the case where all double moves are →↑ or ↑→. But then the number of right moves matches the number of up moves, so our grid is a square grid.

Let us consider the diagonal line joining the lower-left corner to the upper-right corner of the square. All unit squares cut by this diagonal are the same color, say black. By reflection symmetry across this diagonal, the black area under the diagonal is equal to the white area under the diagonal. Note that any →↑ and any ↑→ double move connects two points on this diagonal. Each →↑ double move loses half the area of a black square, and each ↑→ double move gains half the area of a black square. Therefore, the regions above and below the path are balanced if and only if there is an equal number of →↑ and ↑→ moves in our path.

We have reduced the problem to counting the number of paths from the lower left corner to the upper right corner that have the same number of ↑→ double moves as →↑ double moves. We prove that a path has the same number of ↑→ double moves as →↑ double moves if and only if there are exactly m double moves that start with an up arrow and exactly m double moves that end in an up arrow. (Equivalently, there are exactly n double moves that start with a right arrow and exactly n

double moves that end in a right arrow.)

Suppose first that a path has the same number of $\uparrow\rightarrow$ double moves as $\rightarrow\uparrow$ double moves. Note that the $\uparrow\uparrow$ double moves contribute equally to upward moves in the first component and the second component. Since the total of upward moves is $2m$, we must have exactly m double moves that start with an up arrow and exactly m double moves that end in an up arrow.

Conversely, suppose that there are exactly m double moves in our path that start with an up arrow and exactly m double moves that end in an up arrow. Suppose also that we have k $\uparrow\rightarrow$ double moves for some nonnegative integer k. Considering the first components, we see that we have $m-k$ $\uparrow\uparrow$ double moves. Now considering the second components, we get that there are k $\rightarrow\uparrow$ double moves, so the number of $\uparrow\rightarrow$ double moves and the number of $\rightarrow\uparrow$ double moves are the same. This proves our claim.

We have $\binom{m+n}{m}$ choices for the m double moves that start with an up arrow and, independently, $\binom{m+n}{m}$ choices for the m double moves that end in an up arrow. Therefore, the number of up-right paths that divide $\mathcal{R}$ into two domino-tileable subsets equals $\binom{m+n}{m}^2$, as claimed.

Problem 4 $(225; 36, 5, 1, 2, 3, 5, 10, 163)$; *proposed by Hung-Hsun Yu.* Let n be a positive integer, and let $a_0, a_1, \ldots, a_n$ be nonnegative integers such that $a_0 \geq a_1 \geq \cdots \geq a_n$. Prove that

$$\sum_{i=0}^{n} i\binom{a_i}{2} \leq \frac{1}{2}\binom{a_0 + a_1 + \cdots + a_n}{2}. \tag{1}$$

Note: $\binom{k}{2} = \frac{k(k-1)}{2}$ for all nonnegative integers k.

First solution. Let $s = a_0 + a_1 + \cdots + a_n$. Consider the collection of integer lattice points

$$X = \{(i,j) \in \mathbb{Z}^2 \mid 0 \leq i \leq n, 1 \leq j \leq a_i\}$$

in the plane. For points $P = (i_1, j_1)$ and $Q = (i_2, j_2)$ with $i_1 < i_2$ in X, we say that the vector $\overrightarrow{PQ}$ is *ascending* if $j_2 > j_1$, and *descending* if $j_2 < j_1$.

We can count the number of ascending vectors $\overrightarrow{PQ}$ in X by letting $P = (i_1, j_1)$ and $Q = (i_2, j_2)$ with $i_2 = i$ for a given $0 \leq i \leq n$, choosing i_1 with $0 \leq i_1 < i$, and choosing j_1 and j_2 with $1 \leq j_1 < j_2 \leq a_i$. Therefore, the number of ascending vectors in X is equal to $\sum_{i=0}^{n} i\binom{a_i}{2}$.

Note that we have at least as many descending vectors in X as ascending ones: indeed, if $\overrightarrow{PQ}$ is an ascending vector in X with $P = (i_1, j_1)$ and $Q = (i_2, j_2)$ then, since $a_{i_1} \geq a_{i_2}$, the vector $\overrightarrow{P'Q'}$ with $P' = (i_1, j_2)$ and $Q' = (i_2, j_1)$ is a descending vector in X. (See Figure 2 for an illustration.)

Furthermore, the total number of ascending and descending vectors in X is at most the number of pairs of distinct points in X. (In fact, strict inequality holds, unless $s \leq 1$.) This yields $2\sum_{i=0}^{n} i\binom{a_i}{2} \leq \binom{s}{2}$, which establishes (1).

Second solution. Denoting the left side and right side of (1) by L and R, respectively, we establish our inequality by showing that

$$2L \leq \sum_{i=0}^{n} ia_i^2 \leq 2R. \tag{2}$$

The first inequality in (2) is immediate, since replacing $\binom{a_i}{2}$ by $(a_i^2 - a_i)/2$ we get

$$2L = \sum_{i=0}^{n} ia_i^2 - \sum_{i=0}^{n} ia_i \leq \sum_{i=0}^{n} ia_i^2.$$

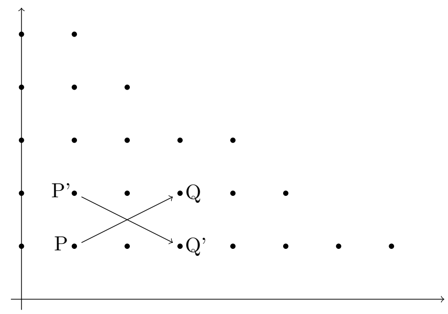

Figure 2: An illustration for the first solution for Problem 4.

Similarly, we have

$$2R = \frac{1}{2}\left(\sum_{i=0}^{n} a_i\right)^2 - \frac{1}{2}\sum_{i=0}^{n} a_i = \frac{1}{2}\sum_{i=0}^{n} a_i^2 + \sum_{0\le j<i\le n} a_j a_i - \frac{1}{2}\sum_{i=0}^{n} a_i \ge \sum_{0\le j<i\le n} a_j a_i.$$

But for $0 \le j < i \le n$ we have $a_j \ge a_i$, so

$$2R \ge \sum_{0\le j<i\le n} a_i^2 = \sum_{i=0}^{n} i a_i^2,$$

as claimed.

Problem 5 $(215; 67, 12, 3, 0, 1, 1, 10, 121)$; *proposed by Carl Schildkraut.* Let H be the orthocenter of acute triangle ABC, let F be the foot of the altitude from C to AB, and let P be the reflection of H across BC. Suppose that the circumcircle of triangle AFP intersects line BC at two distinct points X and Y. Prove that C is the midpoint of XY.

Solution. We start by proving the (well-known) fact that P is on the circumcircle ω of $\triangle ABC$. Indeed, $\angle BAP = \angle FCB$ since they are both complementary angles of $\angle ABC$. Because P is the reflection of H across BC, BC is the angle bisector of $\angle FCP$. Therefore, $\angle BAP = \angle BCP$, implying that P lies on ω, as claimed.

Now let us introduce three additional points: we let O be the center of ω, Q be the antipode of B on ω, and M be the midpoint of QC; see Figure 3.

By Thales's Theorem, QC is perpendicular to BC, and QA is perpendicular to AB. Therefore, AP and QC are parallel, so $AQCP$ is a trapezoid and, since cyclic, it is an isosceles trapezoid. Therefore, $MA = MP$. Furthermore, $AFCQ$ is a right trapezoid and, since M lies on its median that is the perpendicular bisector of AF, we get $MA = MF$. But then M is the center of the circumcircle of $\triangle AFP$, so $MX = MY$; since MC is perpendicular to the line BC, this implies that C is the midpoint of XY, as claimed.

Problem 6 $(189; 98, 10, 13, 6, 7, 10, 13, 32)$; *proposed by Enrique Treviño.* Let S be a set of positive integers with the following properties:

- $\{1, 2, \ldots, 2025\} \subseteq S$.

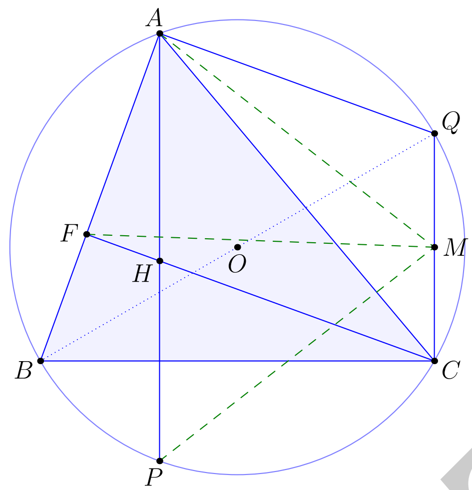

Figure 3: Illustration for Problem 5.

- If $a, b \in S$ and $\gcd(a, b) = 1$, then $ab \in S$.
- If for some $s \in S$, $s + 1$ is composite, then all positive divisors of $s + 1$ are in S.

Prove that S contains all positive integers.

First solution. We will use induction and prove that if all positive integers less than some integer $p \geq 6$ are in S, then p is also in S. Since p is obviously a divisor of itself, by the third property, we may assume that p is prime.

We separate our proof into the following three cases:

- p is neither a Fermat prime nor a Mersenne prime,
- p is a Mersenne prime,
- p is a Fermat prime.

In the first two cases, we prove that $p^2 - 1 \in S$, and in the third case, we establish that $2p - 1 \in S$ or $5p - 1 \in S$. Note that p^2, $2p$, and $5p$ are composite numbers that are divisible by p, so $p \in S$ by the third property in all cases.

Assume first that p is neither a Fermat prime nor a Mersenne prime. In this case we have $p - 1 = 2^\alpha a$ and $p + 1 = 2^\beta b$ for some positive integers α and β and odd integers $a \geq 3$ and $b \geq 3$. Since exactly one of $p - 1$ or $p + 1$ is divisible by 4, we have $\alpha \geq 2$ and $\beta = 1$ or $\beta \geq 2$ and $\alpha = 1$; without loss of generality we assume that the first of these possibilities holds, and we have

$$p^2 - 1 = (p - 1)(p + 1) = 2^{\alpha+1}ab.$$

Here $2^{\alpha+1} < 2^\alpha a = p - 1$, $a < p - 1$, and $b < p - 1$, so by our inductive assumption, $2^{\alpha+1}, a, b \in S$. Furthermore, these three integers are pairwise relatively prime, so by the second property, their product $p^2 - 1 \in S$, completing the proof of our first case.

In the case when p is a Mersenne prime, we have $p - 1 = 2^\alpha$ for some odd integer (indeed, a prime) $\alpha \geq 3$. Then $2^{(\alpha+1)/2} - 1$ and $2^{(\alpha+1)/2} + 1$ are both less than p, so by our inductive assumption, they are in S; since they are also relatively prime, their product $2^{\alpha+1}-1$ is also in S by the second property. The third property then yields that $2^{\alpha+1} \in S$. Now $2^{\alpha-1}+1$ is less than p and is relatively prime to $2^{\alpha+1}$, so by the second property, their product

$$2^{\alpha+1}\left(2^{\alpha-1}+1\right) = 2^{\alpha}\left(2^{\alpha}+2\right) = (p-1)(p+1) = p^2-1$$

is in S. This takes care of the second case.

For our final case, we consider the Fermat prime $p = 2^{2^k}+1$ for some integer $k \geq 2$. Note that $p = \left(2^2\right)^{2^{k-1}}+1 \equiv 2 \bmod 3$, so both $2p-1$ and $5p-1$ are divisible by 3. Let $2p-1 = 3^{\alpha}a$ for some positive integers α and a where a is relatively prime to 3. If $a > 1$, then both 3^{α} and a are less than p and thus are in S; the second property then implies that their product $2p-1$ is in S.

It remains to be considered when $2p-1 = 3^{\alpha}$ for some positive integer α; we then have $\alpha \geq 2$. In this case, $5p-1 = 3p+3^{\alpha} \equiv 6 \bmod 9$. Furthermore, $p = 2^{2^k}+1 = 2^3 \cdot 2^{2^k-3}+1 \equiv 1 \bmod 8$, so $5p-1 \equiv 4 \bmod 8$. Therefore, $5p-1$ is the product of pairwise relatively prime integers 3, 4, and c for some positive integer c, which implies that $5p-1 \in S$. Our proof is now complete.

Second solution. We will use induction to prove that if all positive integers up to 2^k are in S for some $k \geq 4$, then all positive integers up to 2^{k+1} are in S as well.

First, we establish that $2^t \in S$ for all positive integers $t \leq 2k-2$. According to our assumption, $2^{k-1}-1 \in S$ and $2^{k-1}+1 \in S$, so by the second property, their product $2^{2k-2}-1$ is in S. Using the third property, we get that all positive divisors of 2^{2k-2} are in S, as claimed.

Next, we show that all even integers up to 2^{k+1} that are not powers of 2 are in S. Since all such integers are products of a power of 2 and an odd integer, both between 1 and 2^k, the second property yields that they are in S.

Lastly, consider an odd integer n less than 2^{k+1}. In this case, we can write $n-1 = 2^{\alpha}a$ and $n+1 = 2^{\beta}b$ for some positive integers α, β, a, and b, with a and b odd, and 2^{α}, a, and b all at most 2^k, while 2^{β} at most 2^{k+1}. Since $n-1$ and $n+1$ are consecutive even integers, we also know that $\alpha = 1$ or $\beta = 1$ and that a and b are relatively prime. Therefore, n^2-1 is the product of pairwise relatively prime integers $2^{\alpha+\beta}$, a, and b, all of which are in S (even when $\alpha+\beta = k+2$, since $k+2 \leq 2k-2$). The second property then implies that $n^2-1 \in S$, from which the third property yields $n \in S$, which completes our proof.

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Summary. We present the problems and solutions to the 16th Annual United States of America Junior Mathematical Olympiad.

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