A trio of research projects with undergraduates

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LAKE FOREST
COLLEGE

MathFest
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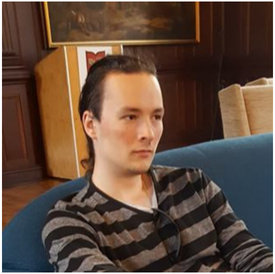
Project 1

On generalizing happy numbers to fractional base number systems with **Mikita Zhylinski**, Lake Forest College.



Inspiring Paper

Happiness is integral but not rational
by Andre Bland, Zoe Cramer, Philip de Castro, Desiree Domini, Tom Edgar, Devon Johnson, Steven Klee, Joseph Koblitz, and Ranjani Sundaresan.
Math Horiz. 25 (2017), no. 1, 8–11.

Happy numbers

- Let $S(n)$ be the sum of the squares of the digits of n.
- Consider iterating S on positive integers.
- The number n, after enough iterations of S, eventually reaches 1 or it eventually reaches the cycle

$$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4.$$

- We call n *happy* if n eventually reaches 1 after enough iterations of S.
- 13 is happy since

$$13 \to 10 \to 1.$$

- Happy numbers are sequence A007770 in OEIS.

TABLE 1. Fixed points and cycles of $S_{2,b}$, $2 \leq b \leq 10$

Base	Fixed Points and Cycles
2	1
3	1, 12, 22 $2 \to 11 \to 2$
4	1
5	1, 23, 33 $4 \to 31 \to 20 \to 4$
6	1 $32 \to 21 \to 5 \to 41 \to 25 \to 45 \to 105 \to 42 \to 32$
7	1, 13, 34, 44, 63 $2 \to 4 \to 22 \to 11 \to 2$ $16 \to 52 \to 41 \to 23 \to 16$
8	1, 24, 64 $4 \to 20 \to 4$ $5 \to 31 \to 12 \to 5$ $15 \to 32 \to 15$
9	1, 45, 55 $58 \to 108 \to 72 \to 58$ $82 \to 75 \to 82$
10	1 $4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$

Fractional Base

For any p/q with $\gcd(p,q) = 1$ and $p > q$, for every positive integer n, there exist *fractional digits* $a_0, a_1, \ldots, a_r$ satisfying $0 \leq a_i < p$ for $i \in \{0, 1, \ldots, r-1\}$ and $0 < a_r < p$ such that

$$n = \sum_{i=0}^{r} a_i \left(\frac{p}{q}\right)^i.$$

We will write

$$n = \overline{a_r a_{r-1} a_{r-2} \ldots a_2 a_1 a_0}_{\frac{p}{q}}.$$

n	n in base $3/2$	n	n in base 3/2
0	$\overline{0}_{\frac{3}{2}}$	6	$\overline{210}_{\frac{3}{2}}$
1	$\overline{1}_{\frac{3}{2}}$	7	$\overline{211}_{\frac{3}{2}}$
2	$\overline{2}_{\frac{3}{2}}$	8	$\overline{212}_{\frac{3}{2}}$
3	$\overline{20}_{\frac{3}{2}}$	9	$\overline{2100}_{\frac{3}{2}}$
4	$\overline{21}_{\frac{3}{2}}$	10	$\overline{2101}_{\frac{3}{2}}$
5	$\overline{22}_{\frac{3}{2}}$	11	$\overline{2102}_{\frac{3}{2}}$

Table: The first 12 non-negative integers in the 3/2 base number system.

Some Results

e	Cycles	n^*
1	(1), (2)	2
2	(1), (5, 8, 9)	8
3	(1), (9), (10), (17, 18)	32
4	(1), (51), (52)	77
5	(1), (131), (98, 99)	185
6	(1), (197, 260, 387, 323, 263, 450), (324, 131, 259)	419
7	(1), (771, 516, 643, 518)	1211
8	(1), (1539, 775, 1284), (1287, 1794, 1796, 2052), (1032), (1033)	2723
9	(1), (2566), (2565)	6557
10	(1), (10247)	13118
11	(1), (14342, 16388, 14344), (14341), (14340)	27968
12	(1), (28678), (28677)	62933

Table: Cycles reached when iterating $S_{e,\frac{3}{2}}$ for different values of e.

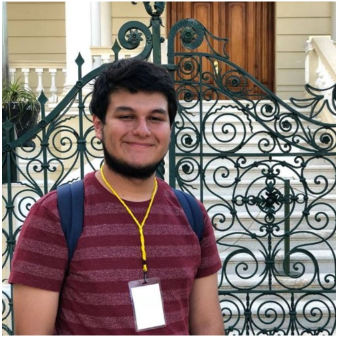
More Results

p/q	$e = 2$	$e = 3$	$e = 4$
5 / 2	(16, 6, 5, 4), (32, 24, 29); $n^* = 39$	(65), (163, 190, 73, 118, 64), (81), (80), (66), (17); $n^* = 239$	(371, 276, 275, 274), (355, 130, 113), (195, 353); $n^* = 1039$
5 / 3	(34, 50), (25), (26), (59), (23), (11), (10); $n^* = 59$	(100, 38, 64, 102, 46), (101, 39), (127, 107, 73, 135), (162), (193), (190, 166, 218), (199, 237); $n^* = 284$	(772, 804, 454, 788, 950, 658, 934, 1126, 1028, 1202, 868, 936, 390), (1027, 1137, 1125), (1122, 994), (1299), (101), (100); $n^* = 1324$
5 / 4	(66, 55), (50), (58, 75, 49, 56, 67), (74, 83), (51); $n^* = 74$	(311, 251, 247, 231, 371), (361), (417), (374), (360), (314), (424, 418, 436, 272, 328, 364); $n^* = 464$	(1786, 1880, 1403, 1594, 1659, 2011, 2075, 1579, 2057, 1947, 1688, 1229, 1641, 1676, 1946, 1673, 1851, 1592, 1419, 1974, 2058, 2012, 2090); $n^* = 2639$
7 / 2	(25, 52), (97); $n^* = 97$	(341, 591, 376, 143, 187, 216, 352, 25, 280, 244, 469, 63, 128, 44, 141, 161, 197, 73, 307, 467, 377, 234, 182, 91), (35), (288, 343, 9, 16, 72), (36), (189), (190), (468); $n^* = 615$	(914, 2065, 1953, 1538, 2819, 2690, 2210, 1507, 1491, 2610, 1856, 1348, 1666, 259, 1808, 2659, 3136, 1824), (1634, 1731, 994), (371, 34, 1313), (130, 354, 289, 1938, 3265, 2930, 1474, 1570), (451, 195, 2177, 1554, 179, 513, 2034, 2530); $n^* = 5417$

Table: Cycles reached when iterating $S_{e,\frac{p}{q}}$, and the value of n^* for different values of e and p/q

Project 2

On a sequence related to the factoradic representation of an integer with **Maximiliano Sánchez Garza**, Universidad Autónoma de Nuevo León.



Inspiring Paper

Sequences of consecutive factoradic happy numbers
by J. Carlson, E. G. Goedhart, and P. E. Harris.
Rocky Mountain J. Math. 50 (2020), 1241–1252.

Factoradic Representation

- Every positive integer n can be written uniquely in the form

$$n = \sum_{i=1}^{k} a_i \cdot i!,$$

  for some positive integer k satisfying $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $1 \leq i \leq k-1$.
- We call this the **factoradic expansion** of n.
- We will use the notation $n = (a_k a_{k-1} \cdots a_1)_!$ to express a number written in its factoradic expansion.
- For example, $8 = 110_!$ because $8 = 0 \cdot 1! + 1 \cdot 2! + 1 \cdot 3!$.

Motivating Result

Let r be a positive integer and define j_r to be the smallest positive integer n satisfying

$$n! > n^{r-1}.$$

Theorem (Carlson, Goedhart, Harris, 2020)

Let r be a positive integer satisfying $2 \leq r \leq 30$. Write n in its factoradic expansion as $n = \sum_{i=1}^{k} a_i i!$ with $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $i \in \{1, 2, \ldots, k-1\}$. Let

$$S_{r,!}(n) = \sum_{i=1}^{k} a_i^r.$$

Then for $n \geq (j_r + 1)!$,

$$S_{r,!}(n) < n.$$

Main Theorem

Theorem

Let r be a positive integer. Write n in its factoradic expansion as $n = \sum_{i=1}^{k} a_i i!$ with $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $i \in \{1, 2, \ldots, k-1\}$. Let

$$S_{r,!}(n) = \sum_{i=1}^{k} a_i^r.$$

Then for $n \geq (j_r + 1)!$,

$$S_{r,!}(n) < n.$$

The sequence j_r

Let r be a positive integer and define j_r to be the smallest positive integer n satisfying

$$n! > n^{r-1}.$$

- The first 20 values of j_r in the On-line Encyclopedia of Integer Sequences are

$$\{2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 27\}.$$

- It is sequence A230319 in OEIS.

Properties of j_r

Some properties of j_r we proved:

- $j_{r+1} - j_r \in \{1, 2\}$ for all r.
- Let $\varepsilon > 0$ be a real number. Then there exists M such that, for integers $r > M$, we have that $j_r < (1 + \varepsilon)r$.
- For a positive integer r, there exists a real number θ_r such that

$$j_r = r + \frac{r}{\log r} + \theta_r \left(\frac{r}{\log r}\right),$$

with $\theta_r \to 0$ as $r \to \infty$.

Project 3

Generalizing Parking Functions with Randomness with **Melanie Tian**, Lake Forest College.

Inspiring Paper

Parking functions: choose your own adventure
Joshua Carlson, Alex Christensen, Pamela E. Harris, Zakiya Jones, and Andrés Ramos Rodríguez.
College Math. J. 52 (2021), no. 4, 254–264.

Parking Functions

- Consider n cars $C_1, C_2, \ldots, C_n$ that want to park in a parking lot with parking spaces $1, 2, \ldots, n$ that appear in order.
- Each car C_i has a parking preference $\alpha_i \in \{1, 2, \ldots, n\}$.
- The cars appear in order, if their preferred parking spot is not taken, they take it, if the parking spot is taken, they move forward until they find an empty spot. If they don't find an empty spot, they don't park.
- An n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is said to be a parking function, if this list of preferences allows every car to park under this algorithm.
- For example $(2, 1, 1, 2)$ is a parking function while $(4, 3, 3, 1)$ is not.

Theorem (Konheim, Weiss, 1966)

Given a positive integer n, The number of parking functions is

$$(n+1)^{n-1}.$$

Naples parking

Consider the following variant, called Naples-parking:

- If a car is parked in C_i's preferred spot, then C_i will check if the previous spot is taken, if not, he takes that spot, otherwise C_i continues forward.

Theorem (Christensen, Harris, Jones, Loving, Ramos Rodríguez, Rennie, Rojas Kirby, 2020)

The number $N(n+1)$ of Naples parking functions of length $n+1$ is counted recursively by

$$N(n+1) = \sum_{i=0}^{n} \binom{n}{i} \min(i+2, n+1) N(i)(n-i+1)^{n-i-1}.$$

Variant introducing randomness

Suppose we consider Naples parking, but instead of a car moving the one space backward automatically, they decide with probability p to take one space back or just stay in the spot.
Some examples with $p = 1/2$.

- The tuple $(2, 1, 1, 2)$ has probability 1 of parking.
- The tuple $(4, 3, 3, 1)$ has probability 1/2 of parking.

Generalizing the recursion formula

Theorem (CHJLR-RRR-K, 2020)

If k, n are nonnegative integers with $k < n$, then the number $N_k(n)$ of k-Naples parking functions of length n is counted recursively by

$$N(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} N(i)(n-i)^{n-i-2} \min(i+2, n).$$

Theorem

Let $T_p(n)$ be the expected number of parking preferences.

$$T_p(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} T_p(i)(n-i)^{n-i-2}(i+1+p\min\{1, n-i-1\})$$

Do you see a Pattern?

p	0	1/64	2/64	3/64	4/64	5/64	6/64	7/64
$f(p)$	339472	1	136	1	2194	1	209	1
p	8/64	9/64	10/64	11/64	12/64	13/64	14/64	15/64
$f(p)$	12466	1	140	1	3107	1	143	1
p	16/64	17/64	18/64	19/64	20/64	21/64	22/64	23/64
$f(p)$	40610	1	141	1	1361	1	74	1
p	24/64	25/64	26/64	27/64	28/64	29/64	30/64	31/64
$f(p)$	14253	1	75	1	1589	1	148	1
p	32/64	33/64	34/64	35/64	36/64	37/64	38/64	39/64
$f(p)$	94792	1	30	1	1171	1	33	1
p	40/64	41/64	42/64	43/64	44/64	45/64	46/64	47/64
$f(p)$	4861	1	104	1	576	1	37	1
p	48/64	49/64	50/64	51/64	52/64	53/64	54/64	55/64
$f(p)$	35324	1	35	1	614	1	38	1
p	56/64	57/64	58/64	59/64	60/64	61/64	62/64	63/64
$f(p)$	6819	1	39	1	734	1	42	1

Table: Distribution of probability for $n = 7$, p for probability and $f(p)$ for number of preferences of probability p.

Fun Theorems

Theorem

There is one and only one parking preference for which the probability that every car parks is $\frac{2t-1}{2^{n-1}}$*, where* $t \in [1, 2^{n-2}]$.

Theorem

The condition of having probability $\frac{t}{2^{n-1}}, t \in \{0, 1, \ldots, 2^{n-1}\}$ *of success all have at least 1 preference satisfying.*

Thank you

Thank you