# A problem from the Mexican Mathematical Olympiad that led to some research articles

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Dedicated to Florian Luca

# The Problem

$31^a$ Olimpiada Mexicana de Matemáticas
Concurso Nacional

Santiago, Nuevo León, 2017
Primer día

1. En un tablero de ajedrez de $2017 \times 2017$, se han colocado en la primera columna 2017 caballos de ajedrez, uno en cada casilla de la columna. Una tirada consiste en elegir dos caballos distintos y de manera simultnea moverlos como se mueven los caballos de ajedrez. Encuentra todos los posibles valores enteros de $k$ con $1 \leq k \leq 2017$, para los cuales es posible llegar a travs de varias tiradas, a que todos los caballos estn en la columna $k$, uno en cada casilla.

   **Nota.** Un caballo se mueve de una casilla $X$ a otra $Y$, solamente si $X$ y $Y$ son las esquinas opuestas de un rectngulo de $3 \times 2$ o de $2 \times 3$.

2. Un conjunto de $n$ números enteros positivos distintos es *equilibrado*, si el promedio de cualesquiera $k$ números del conjunto es un número entero, para toda $1 \leq k \leq n$. Encuentra la mayor suma que pueden tener los elementos de un conjunto equilibrado, con todos sus elementos menores o iguales que 2017.

3. Sea $ABC$ un triángulo acutángulo cuyo ortocentro es el punto $H$. La circunferencia que pasa por los puntos $B$, $H$ y $C$ vuelve a intersectar a las rectas $AB$ y $AC$ en los puntos $D$ y $E$, respectivamente. Sean $P$ y $Q$ los puntos de intersección de $HB$ y $HC$ con el segmento $DE$, respectivamente. Se consideran los puntos $X$ e $Y$ (distintos de $A$) que están sobre las recta $AP$ y $AQ$, respectivamente, de manera que los puntos $X$, $A$, $H$ y $B$ están sobre un círculo y los puntos $Y$, $A$, $H$ y $C$ están sobre un círculo. Muestra que las rectas $XY$ y $BC$ son paralelas.

**Problem 2, 31st Mexican Mathematical Olympiad, November 2017**

A set with $n$ distinct positive integers is said to be *balanced* if the mean of any $k$ numbers in the set is an integer, for any $1 \leq k \leq n$. Find the largest possible sum of the elements of a balanced set with all numbers in the set less than or equal to 2017.

# Sketch of solution

- Consider a balanced set with $n$ elements. Say $S = \{a_1, a_2, \ldots, a_n\}$.
- Let $k \leq n - 1$. Note that by fixing any $k - 1$ terms, the $k$-th term has to be of the same congruence modulo $k$ for any other number. Therefore, they are all congruent modulo $k$.
- Since $a_i \equiv a_j \mod k$ for all pairs $i, j$ and all $k \leq n - 1$, then all the numbers are congruent modulo $M = \mathrm{lcm}\{1, 2, \ldots, n - 1\}$.
- Note that if $n \geq 8$, then a balanced set consists of elements congruent to $\mathrm{lcm}\{1, 2, \ldots, 7\} = 420$. Since we can't have 8 positive integers $\leq 2017$ congruent to each other modulo 420, then we need to consider balanced sets with at most 7 elements.
- $S = \{2017, 2017 - 60, \ldots, 2017 - 6 \cdot 60\}$ is the balanced set with 7 elements of maximal sum (12859). If you have 6 elements or less the sum is at most $6 \cdot 2017 < 12859$.

# Slight variant

Consider the same problem but with numbers $\leq 3000$ instead of $\leq 2017$. What happens?

- Since $420 \cdot 7 \leq 3000$, we can fit an 8-element balanced set, namely $\{3000, 3000 - 420, \ldots, 3000 - 7 \cdot 3000\}$. The sum of the elements of this set is 12240.
- The 7-element balanced set $\{3000, 3000 - 60, \ldots, 3000 - 6 \cdot 60\}$ has sum 19740.
- The 7-element balanced set has a higher sum than the 8-element balanced set!

# Generalization

- For a positive integer $N$, let $M(N)$ be the size of the largest balanced set all of whose elements are $\leq N$.
- Let $S(N)$ be the size of the set with maximal sum among balanced sets all of whose elements are $\leq N$.

## For what $N$ is $M(N) = S(N)$?

For example $M(2017) = S(2017)$, yet $M(3000) \neq S(3000)$.

# Numerics

Using a computer, we can verify that if $N \leq 1000000$, then $M(N) = S(N)$ for

$$\begin{aligned} 1 &\leq N \leq 18 \\ 31 &\leq N \leq 48 \\ 85 &\leq N \leq 300 \\ 571 &\leq N \leq 2940 \\ 18481 &\leq N \leq 22680 \\ 54181 &\leq N \leq 304920 \end{aligned}$$

# Pattern

Consider 18, 48, 300, 2940, 22680, 304920. Let

$$L(n) = \operatorname{lcm}\{1, 2, \ldots, n\}.$$

Then

$$\begin{aligned} 18 &= 3L(3) \\ 48 &= 4L(4) \\ 300 &= 5L(5) \\ 2940 &= 7L(7) \\ 22680 &= 9L(9) \\ 304920 &= 11L(11) \end{aligned}$$

# Theorems about $mL(m)$

**Theorem**

*Let $p$ be prime. Then $M(pL(p)) = S(pL(p))$. Furthermore, $M(pL(p)+1) \neq S(pL(p)+1)$.*

**Theorem**

*If $m$ is not a prime power, then $M(mL(m)) \neq S(mL(m))$.*

# Stronger statements

**Theorem**

*For $m \geq 10^{10^{15}}$ of the form $q^k$ for a prime $q$ and an exponent $k \geq 3$, then $M(mL(m)) \neq S(mL(m))$.*

Using results from Carneiro, Milinovich, and Soundararajan (2019) on large prime gaps assuming the Generalized Riemann Hypothesis (GRH), we can prove

**Theorem**

*Assuming GRH, if $m = q^k$ for a prime $q$ and exponent $k \geq 3$, then $M(mL(m)) \neq S(mL(m))$.*

**Conjecture**

$$S(mL(m)) = M(mL(m))$$

*if and only if $m$ is prime or $m \in \{4, 9, 121\}$.*

# Density Question

- Let $A$ be the set of all $N$ for which $S(N) = M(N)$.
- Let $A(x)$ be the set of all $N \leq x$ for which $S(N) = M(N)$.

Does $\lim_{x\to\infty} \dfrac{A(x)}{x}$ exist?

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Does $\lim_{x\to\infty} \frac{A(x)}{x}$ exist?

## Theorem

*Let $\delta^+$ and $\delta^-$ be the upper and lower natural densities, i.e.,*

$$\delta^+ = \limsup_{x\to\infty} \frac{A(x)}{x},$$

$$\delta^- = \liminf_{x\to\infty} \frac{A(x)}{x}.$$

*Then*

$$\delta^+ = 1, \qquad \delta^- = 0.$$

*Therefore $\lim_{x\to\infty} \frac{A(x)}{x}$ does not exist.*

# Logarithmic density

- When these results were presented at the Integers conference in May 2025, Alexander Borisov asked me, what about the logarithmic density?
- Let's define logarithmic density first. The logarithmic density of a set $A$ (if it exists) is

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{a\in A(x)} \frac{1}{a}.$$

## Theorem

*Suppose a set $S$ of positive integers has natural density $\delta$. Then the logarithmic density exists and it equals $\delta$.*

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**Theorem**

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**Theorem**

*The set of values of $N$ such that $M(N) = S(N)$ has logarithmic density 0.*

# Thank you

# Examples of Logarithmic Density

- Suppose we want to find the logarithmic density of the set of even integers.

$$\sum_{\substack{n\le x\\ 2|n}} \frac{1}{n} = \frac{1}{2}\sum_{k\le x/2}\frac{1}{k} = \frac{1}{2}\log(x/2) + O(1) = \frac{1}{2}\log x + O(1),$$

  so the logarithmic density is 1/2.
- What about the logarithmic density of perfect squares?

$$\sum_{k\le\sqrt{x}} \frac{1}{k^2} = O(1),$$

  so the logarithmic density of perfect squares is 0.

# Benford's Law

- Let $D_1$ be the number of positive integers whose leading digit is 1. It turns out the natural density of this set does not exist.

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- The logarithmic density of $D_1$ is

$$\frac{\log 2}{\log 10}.$$

- We can find the logarithmic density for $D_i$, the number of positive integers with leading digit $i$. The density is

$$\frac{\log i - \log{(i-1)}}{\log 10}.$$