Math 311: Introduction to Real Analysis
Bonus Homework

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Problem 1. (Exercise **4.2.1**) Prove that if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^3$ converges.

Problem 2. (Exercises **4.2.2** and **4.2.3**) Consider the series

$$1 + \frac{1}{2\sqrt[3]{2}} + \frac{1}{2\sqrt[3]{2}} - \frac{1}{\sqrt[3]{2}} + \frac{1}{3\sqrt[3]{3}} + \frac{1}{3\sqrt[3]{3}} + \frac{1}{3\sqrt[3]{3}} - \frac{1}{\sqrt[3]{3}}$$
$$+ \cdots + \underbrace{\frac{1}{n\sqrt[3]{n}} + \frac{1}{n\sqrt[3]{n}} + \cdots + \frac{1}{n\sqrt[3]{n}}}_{n \text{ times}} - \frac{1}{\sqrt[3]{n}} + \cdots.$$

(a) Show that the series converges.

(b) Show that if b_k is the kth summand of the series, then $\sum_{i=1}^{\infty} b_i^3$ diverges.

(c) Why doesn't this contradict Problem 1?

Problem 3. (Exercise **3.4.29**) Suppose that f is continuous on $[a, \infty)$ and $\lim_{x\to\infty} f(x)$ is finite. Show that f is bounded on $[a, \infty)$.

Problem 4. In class, the following theorem (Theorem 3.13: L'Hospital's Rule ∞/∞) was proved:

If f and F are both differentiable at every point except $x = a$ in an open interval that contains a, if

$$\lim_{x\to a} |F(x)| = \infty,$$

if $F'(x) \neq 0$ for all x in this open interval, and if $\lim_{x\to a} f'(x)/F'(x)$ exists and is finite, then

$$\lim_{x\to a} \frac{f(x)}{F(x)} = \lim_{x\to a} \frac{f'(x)}{F'(x)}. \tag{1}$$

Suppose that $\lim_{x\to a} f'(x)/F'(x) = \infty$. Show (1) is also true under this condition.

Problem 5. (Exercise **4.2.13**) Consider the following two series:

$$\frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^5} + \frac{1}{2^6} + \cdots + \left(\frac{5-(-1)^n}{2}\right)^{-n} + \cdots. \tag{2}$$

$$\frac{1}{2} + 2^2 + \frac{1}{2^3} + 2^4 + \frac{1}{2^5} + 2^6 + \cdots + 2^{(-1)^n n} + \cdots. \tag{3}$$

(a) Use the ratio test to determine whether (2) is convergent. Note that the test could be inconclusive.

(b) Use the root test to determine whether (2) is convergent. Note that the test could be inconclusive.

(c) Use the ratio test to determine whether (3) is convergent. Note that the test could be inconclusive.

(d) Use the root test to determine whether (3) is convergent. Note that the test could be inconclusive.

Problem 6. (Exercises **4.2.8** and **4.2.9**) Given a series $a_1 + a_2 + a_3 + \cdots$, assume that we can find a bound α and a subscript N such that $n \geq N$ implies that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \alpha.$$

(a) Prove that given any $\epsilon > 0$, there is a subscript M such that $n \geq M$ implies that

$$\sqrt[n]{|a_n|} < \alpha + \epsilon.$$

(b) Show that this does not necessarily imply that $\sqrt[n]{|a_n|} \leq \alpha$ for some n large enough.

(c) Prove that if the ratio test tells us that our series converges absolutely, then the root test will also tell us that our series converges absolutely.

Problem 7. (Exercises **4.2.10**, **4.2.11**, and **4.2.12**) Given a series $a_1 + a_2 + a_3 + \cdots$, assume that we can find a bound β and a subscript N such that $n \geq N$ implies that

$$\left|\frac{a_{n+1}}{a_n}\right| \geq \beta.$$

(a) Prove that for any $\epsilon > 0$, there exists a subscript M such that for $n \geq M$ we have

$$\sqrt[n]{|a_n|} > \beta - \epsilon.$$

(b) Prove that if $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists, then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(c) Find an infinite series of positive summands for which the root test shows it diverges, but the ratio test is inconclusive.

(d) Explain why the example from part (c) does not contradict part (b).

Problem 8. Suppose $a_1 + a_2 + a_3 + \cdots$ converges conditionally. Prove

(a) The sum of the positive terms of a_i diverges to infinity.

(b) The sum of the negative terms of a_i diverges to negative infinity.

(c) You can rearrange the terms in the terms of $a_1 + a_2 + \cdots$ so that the series converges to 1.

(d) Let x be any real number. Show you can rearrange the terms of $a_1 + a_2 + \cdots$ to converge to x.