# Homework 1
# Math 311: Introduction to Real Analysis

August 29, 2017

## 1 Chapter 2

**Problem 1.** (**Exercise 2.1.2**) Archimedes' formula for the area of a parabolic region is obtained constructing triangles where the base is the line segment that bounds the region and the apex is located at the point where the tangent line to the parabola is parallel to the base. Show that the tangent to $y = 1 - x^2$ at

$$\left( \frac{k + \frac{1}{2}}{2^n}, 1 - \left( \frac{k + \frac{1}{2}}{2^n} \right)^2 \right)$$

has the same slope as the line segment connecting the two endpoints:

$$\left( \frac{k}{2^n}, 1 - \frac{k^2}{2^{2n}} \right) \quad \text{and} \quad \left( \frac{k+1}{2^n}, 1 - \frac{(k+1)^2}{2^{2n}} \right).$$

**Problem 2.** (**Exercise 2.1.3**) Show that if we take the parabolic region and inscribe a triangle whose base is the line segment that bounds the region and whose apex is located at the point where the tangent line to the parabola is parallel to the base, then the area of the triangle is more than half the area of the parabolic region.

**Problem 3.** (**Exercise 2.1.7**) Consider the series

$$3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots + \frac{(-1)^k 3}{2^k} + \cdots, \quad k \geq 0.$$

(a) Find the target value, $T$, of the partial sums.

(b) How many terms do you have to take in order to guarantee that all of the partial sums from that point on will be smaller than $M = T + \frac{1}{10}$?

(c) How many terms do you have to take in order to guarantee that all of the partial sums from that point on will be larger than $L = T - \frac{1}{10}$?

(d) How many terms do you have to take in order to guarantee that all of the partial sums from that point on will be within $\frac{1}{100}$ of $T$?

**Problem 4.** (**Exercise 2.1.8**) Consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{k-1}}{k} + \cdots.$$

(a) Explain why there should be a target value $T$. You may not be able to prove that $T = \ln(2)$, but you should still be able to explain why there should be one.

(b) How many terms will be enough to guarantee that all of the partial sums from that point on will be within $\frac{1}{10}$ of $T$? Explain the reasoning that leads to this answer.

**Problem 5.** (**Exercise 2.2.2**) Consider the series

$$1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots + (-1)^k \frac{3^k}{4^k} + \cdots.$$

(a) Find the target value of the series.

(b) Find a value of $n$ such that any partial sum with at least $n$ terms is within 0.001 of the target value. Justify your answer.

**Problem 6.** (**Exercise 2.2.4**) Consider the series

$$1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots + \frac{1}{2^{3k}} + \frac{1}{2^{3k+1}} - \frac{1}{2^{3k+2}} + \cdots.$$

(a) Find the target value of the series.

(b) Find a value of $n$ such that any partial sum with at least $n$ terms is within 0.001 of the target value. Justify your answer.

**Problem 7.** (**Exercise 2.2.5**) It is tempting to differentiate each side of the geometric series

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{(when } |x| < 1\text{)}$$

to assert that

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}.$$

Following Cauchy's advice, we know we need to be careful.

(a) Find the difference between $1 + 2x + 3x^2 + \ldots + nx^{n-1}$ and $\frac{1}{(1-x)^2}$.
**Hint:** Differentiate each side of $1 + x + x^2 + \cdots + x^n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$.

(b) For which values of $x$ will the difference approach 0 as $n$ increases? Justify your answer.

**Problem 8.** (**Exercise 2.2.6**) Consider the series

$$1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \cdots + (-1)^{k-1} \frac{k}{3^{k-1}} + \cdots.$$

(a) Find the target value of the series.

(b) Find a value of $n$ such that any partial sum with at least $n$ terms is within 0.001 of the target value. Justify your answer.