Homework 3
Math 311: Introduction to Real Analysis

September 19, 2017

Problem 1. (**Exercise 2.6.2 and 2.6.3**)

(a) Find a power series in x that would imply that

$$1 - 1 + 1 - 1 + \cdots = \frac{4}{7}$$

when x is set equal to 1.

(b) Given any nonzero integers m and n, find a power series in x that would imply that

$$1 - 1 + 1 - 1 + \cdots = \frac{m}{n}$$

when x is set equal to 1.

Problem 2. (**Exercise 3.1.2 (a,b,d)**) Find the derivatives (where they exist) of the following functions. The function denoted by $\lfloor x \rfloor$ sends x to the greatest integer less than or equal to x. For example $\lfloor 3.1 \rfloor = 3, \lfloor 2 \rfloor = 2, \lfloor 2.7 \rfloor = 2, \lfloor -3.1 \rfloor = -4$.

(a) $f(x) = x|x|$, for $x \in \mathbb{R}$.

(b) $f(x) = \sqrt{|x|}$, for $x \in \mathbb{R}$.

(d) $f(x) = (x - \lfloor x \rfloor)\sin^2(\pi x)$, for $x \in \mathbb{R}$.

Problem 3. (**Exercise 3.1.5**) Show that the function given by

$$f(x) = \begin{cases} x^2 \left|\cos\left(\frac{\pi}{x}\right)\right|, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is not differentiable at $x_n = \frac{2}{2n+1}$, for n an integer, but is differentiable at 0.

Problem 4. (**Exercise 3.1.7**) Let $f(x) = x^2, f'(a) = 2a$. Let $E(x, a)$ be defined as

$$E(x, a) = f'(a) - \frac{f(x) - f(a)}{x - a}.$$

(a) Find the error $E(x, a)$ in terms of x and a.

(b) How close must x be to a if $|E(x, a)|$ is to be less than 0.01?

(c) How close must x be to a if $|E(x, a)|$ is to be less than 0.0001?

Problem 5. (**Exercise 3.1.10**) Let $f(x) = \sin(x)$.

(a) Find the error $E(x, \pi/2)$ as a function of x.

(b) Graph $E(x, \pi/2)$.

(c) Find a δ to respond with if you are given $\epsilon = 0.1$.

(d) Find a δ to respond with if you are given $\epsilon = 0.0001$.

(e) Find a δ to respond with if you are given $\epsilon = 10^{-100}$.

Problem 6. (**Exercise 3.1.11**) Use the definition of differentiability to prove that $f(x) = |x|$ is not differentiable at $x = 0$, by finding an ϵ for which there is no δ response. Explain your answer.

Problem 7. (**Exercise 3.1.12**)

(a) Graph the function $f(x) = x \sin\left(\frac{1}{x}\right)$ (with $f(0) = 0$) for $-2 \leq x \leq 2$.

(b) Prove that $f(x)$ is not differentiable at $x = 0$, by finding an ϵ for which there is no δ response. Explain your answer.

Problem 8. (**Exercise 3.2.1**) Where does Cauchy's proof of the mean value theorem break down if we try to apply it to the function defined by $f(x) = x \sin(1/x)$ ($f(0) = 0$) over the interval $[0, 1]$. Note: The mean value theorem does apply to this function, but Cauchy's approach cannot be used to establish this fact.