Homework 4
Math 311: Introduction to Real Analysis

October 5, 2017

Problem 1. (**Exercise 3.3.1**) Prove that the function defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$ by finding an ϵ for which there is no reply.

Problem 2. (**Exercise 3.3.6**) At what values of x is the function f continuous? Justify your answer.

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational or } x = 0 \\ \frac{qx}{q+1} & \text{if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ are relatively prime integers with } q > 0. \end{cases}$$

Problem 3. (**Exercise 3.3.8**) Let f be a continuous function from $[0, 1]$ to $[0, 1]$. Show that there must be an x in $[0, 1]$ for which $f(x) = x$.

Problem 4. (**Exercise 3.3.15**) Let

$$f(x) = \lfloor x \rfloor + (x - \lfloor x \rfloor)^{\lfloor x \rfloor},$$

for $x \geq 1/2$.

(a) Show that f is continuous on any $a \geq 1/2$.

(b) Show that f is strictly increasing on $[1, \infty)$.

Problem 5. (**Exercise 3.3.28**) Consider the function that takes the tenths digit in the decimal expansion[1] of x and replaces it with a 1. For example $f(2.57) = 2.17, f(3) = 3.1, f(\pi) = 3.14159\ldots = \pi$.

(a) Where is this function continuous? Justify your answer.

(b) Where is this function discontinuous? Justify your answer.

Problem 6. (**Exercise 3.4.6**) Find the greatest lower bound (infimum) and the least upper bound (supremum) of the following sets.

(a) the interval $(0, 3)$.

(b) $\{1, 1/2, 1/4, 1/8, \ldots\}$.

(c) $\{1, 1 + 1/2, 1 + 1/2 + 1/4, 1 + 1/2 + 1/4 + 1/8, \ldots\}$.

(e) $\{0.2, 0.22, 0.222, 0.2222, \ldots\}$.

[1] The decimal expansion of a number is not unique since $3 = 2.999\cdots$. In this question, we assume that you don't allow infinite 9's at the end in your decimal expansion. With that choice, the decimal expansion becomes unique and the function is well-defined.

Problem 7. (**Exercise 3.4.6**) Find the greatest lower bound (infimum) and the least upper bound (supremum) of the following sets.

(i) $\{\frac{m}{n} \mid m, n \in \mathbb{N}\}$.

(j) $\{\sqrt{n} - \lfloor \sqrt{n} \rfloor \mid n \in \mathbb{N}\}$.

(k) $\{x \mid x^2 + x + 1 > 0\}$.

(l) $\{x + \frac{1}{x} \mid x > 0\}$.

Problem 8. (**Exercise 3.4.11**) Prove that if "every set with an upper bound has a least upper bound," then the nested interval principle holds.