Homework 5
Math 311: Introduction to Real Analysis

October 30, 2017

Problem 1. (**Exercise 3.4.1**) Give an example of a function that exists and is bounded for all x in the interval $[0, 1]$ but which never achieves either its least upper bound or its greatest lower bound over this interval.

Problem 2. (**Exercise 3.4.7**) Prove that for any set S, the negative of the supremum of $-S = \{-s \mid s \in S\}$ is a lower bound for S and that there's no larger lower bound (i.e., that it is the infimum).

Problem 3. (**Exercise 3.4.12 and Exercise 3.4.13**)

(a) Use the existence of a least upper bound for any bounded set to prove that if $g'(x) > 0$ for all $x \in [a, b]$, then g is increasing over $[a, b]$ ($a \leq x_1 < x_2 \leq b$ implies that $g(x_1) < g(x_2)$).

(b) Prove that if $f'(x) \geq 0$ for all $x \in [a, b]$ and if $a \leq x_1 < x_2 \leq b$, then $f(x_1) \leq f(x_2)$.

Problem 4. (**Exercise 3.4.22**) Let $P(x)$ be any polynomial of degree at least 2, all of whose roots are real and distinct. Prove that all of the roots of $P'(x)$ must be real.

Problem 5. (**Exercise 3.4.28**) Let f be differentiable on $[a, b]$ such that $f(a) = 0 = f(b)$ and $f'(a) > 0, f'(b) > 0$. Prove that there is at least one $c \in (a, b)$ for which $f(c) = 0$ and $f'(c) \leq 0$.

Problem 6. (**Exercise 3.5.4**) Show that each of the following equations has exactly one real root.

(a) $x^{13} + 7x^3 - 5 = 0$

(b) $3^x + 4^x = 5^x$.

Problem 7. (**Exercise 3.5.7 and 3.5.8**) In the following two evaluations explain what is wrong with the application of L'Hospital's rule:

(a) Using L'Hospital's rule we get the following limit evaluation:

$$\lim_{x \to 0} \frac{3x^2 - 1}{x - 1} = \lim_{x \to 0} \frac{6x}{1} = 0.$$

However, the limit is actually 1. How was L'Hospital misused?

(b) Let $f(x) = x^2 \sin(1/x)$ and $F(x) = x$. Each of these functions approaches 0 as x approaches 0, so by L'Hospital's rule

$$\lim_{x \to 0} \frac{f(x)}{F(x)} = \lim_{x \to 0} \frac{f'(x)}{F'(x)} = \lim_{x \to 0} \frac{2x \sin(1/x) - \cos(1/x)}{1},$$

which does not exist. However, the limit of $f(x)/F(x)$ as $x \to 0$ is 0. What went wrong?

Problem 8. (**Exercise 3.5.16**) Let $f(x) = x^{1/x}$ for $x > 0$.

(a) Prove

$$\lim_{x \to \infty} x^{1/x} = 1.$$

Hint: Use L'Hospital to calculate $\lim_{x \to \infty} \ln(x^{1/x})$.

(b) What is the maximum of $f(x)$?