Homework 6
Math 311: Introduction to Real Analysis

October 31, 2017

Problem 1. (**Exercise 3.5.3**) For $x > -1, x \neq 0$, show that

$$\begin{aligned} (1+x)^\alpha &> 1 + \alpha x \qquad \text{if } \alpha > 1 \text{ or } \alpha < 0, \\ (1+x)^\alpha &< 1 + \alpha x \qquad \text{if } 0 < \alpha < 1. \end{aligned}$$

Problem 2. (**Exercise 3.5.17**) Let f and g be functions with continuous second derivatives on $[0,1]$ such that $g'(x) \neq 0$ for $x \in (0,1)$ and $f'(0)g''(0) - f''(0)g'(0) \neq 0$. Define a function θ for $x \in (0,1)$ so that $\theta(x)$ is one of the values that satisfies the generalized mean value theorem,

$$\frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(\theta(x))}{g'(\theta(x))}.$$

Show that

$$\lim_{x \to 0^+} \frac{\theta(x)}{x} = \frac{1}{2}.$$

Problem 3. (**Exercise 3.5.19**) Suppose f is differentiable on $[a,b]$. Define g in terms of f as follows:

$$g(x) = \begin{cases} f'(a), & x = a, \\ \frac{f(2x-a)-f(a)}{2x-2a}, & a < x \leq \frac{a+b}{2}, \\ \frac{f(b)-f(2x-b)}{2b-2x}, & \frac{a+b}{2} \leq x < b, \\ f'(b), & x = b. \end{cases}$$

Prove g is continuous.

Problem 4. (**Exercise 4.1.1**) Consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

(a) How many terms for we need to take to be within $\epsilon = .0001$ of the target value 2?

(b) How many terms for we need to take to be within $\epsilon = 10^{-1000000}$ of the target value 2?

Problem 5. (**Exercise 4.1.4**) Consider the series

$$1 + \sum_{k=1}^{n} \frac{k!}{100^k}.$$

(a) Evaluate the partial sums for the multiples of 10 up to $n = 400$.

(b) Describe and discuss what you see happening.

Problem 6. (**Exercise 4.1.13**) Calculate the partial sums

$$S_n = \sum_{k=2}^{n} \frac{\sin(k/100)}{\ln(k)}$$

up to at least $n = 2000$.

(a) Describe what you see happening.

(b) Make a guess of the approximate value to which this series is converging.

(c) Explain the rationale behind your guess.

Problem 7. (**Exercise 4.1.16**) Let

$$\epsilon_n = \begin{cases} 1 & \text{for } 2^{2k} \leq n < 2^{2k+1}, \\ -1 & \text{for } 2^{2k+1} \leq n < 2^{2k+2}, \end{cases}$$

where $k = 0, 1, 2, \ldots$. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{n}$$

converges absolutely, converges conditionally, or diverges.

Problem 8. (**Exercise 4.2.4 (a,b,c,d)**) For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges.

(a)

$$\frac{\arctan(1)}{2} + \frac{\arctan(2)}{2^2} + \cdots + \frac{\arctan(n)}{2^n} + \cdots$$

(b)

$$1 + \frac{1}{4} + \frac{2^2}{4^2} + \cdots + \frac{n^2}{4^n} + \cdots$$

(c)

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

(d)

$$\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \cdots + (-1)^{n-1} \frac{1}{n(n+1)} + \cdots$$