Report on the 54th Annual USA Mathematical Olympiad

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The USA Mathematical Olympiad (USAMO) is the final round in the American Mathematics Competitions series for high school students, organized each year by the Mathematical Association of America. The competition follows the style of the International Mathematical Olympiad (IMO): it consists of three problems each on two consecutive days, with an allowed time of four and a half hours both days.

The 54th annual USAMO was given on Tuesday, March 19 and Wednesday, March 20, 2025, and was taken by 308 students. Further information on the American Mathematics Competitions program can be found on the site https://maa.org/student-programs/amc/. Below we present the problems and solutions of the competition; a similar article for the USA Junior Mathematical Olympiad (USAJMO), offered to students in grade 10 or below, can be found in a concurrent issue of the *College Mathematics Journal*.

The problems of the USAMO are chosen—from a large collection of proposals submitted for this purpose—by the USAMO/USAJMO Editorial Board, under the leadership of co-editors-in-chief Oleksandr Rudenko and Enrique Treviño. This year's problems were created by John Berman, Cheng-Yin Chang, Carl Schildkraut, and Hung-Hsun Yu.

The solutions presented here are those of the present authors, relying in part on the submissions of the problem authors. Each problem was worth 7 points; the nine-tuple

$$(n; a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$$

states the number of students who submitted a paper for the relevant problem, followed by the numbers who scored $0, 1, \ldots, 7$ points, respectively.

Problem 1 $(308; 58, 39, 12, 9, 1, 11, 17, 161)$; *proposed by John Berman.* Let k and d be positive integers. Prove that there exists a positive integer N such that for every odd integer $n > N$, the digits in the base-$2n$ representation of n^k are all greater than d.

Solution. We will prove that $N = 2^{k-1}(d+1)$ satisfies the requirement.

Let us first exhibit the base-$2n$ representation of n^k. Let $i = 0, 1, \dots, k$. We start by letting q_i and r_i be the quotient and the remainder of n^k when divided by $(2n)^i$; to be more explicit, we can write $n^k = q_i(2n)^i + r_i$ where $q_i = \lfloor n^k/(2n)^i \rfloor$ and r_i is an integer with $0 \le r_i < (2n)^i$. Observe that the sequence $r_0, \dots, r_k$ is non-decreasing with $r_0 = 0$ and $r_k = n^k$. In terms of these notations, we can express the i-th digit (with $i = 0, 1, \dots, k-1$ counting from the right) in the base-$2n$ representation of n^k as

$$d_i = \frac{r_{i+1} - r_i}{(2n)^i}.$$

To verify, note that d_n is an integer since $r_{i+1} - r_i$ is divisible by $(2n)^i$, $0 \le d_i < 2n$, and

$$\sum_{i=0}^{k-1} d_i(2n)^i = \sum_{i=0}^{k-1}(r_{i+1} - r_i) = r_k - r_0 = n^k.$$

Our next goal is to find a lower bound for d_i. For that purpose, we compute

$$n^k - r_i - d_i(2n)^i = (n^k - r_{i+1}) + (r_{i+1} - r_i - d_i(2n)^i) = q_{i+1}(2n)^{i+1}.$$

Since the right side and n^k are both divisible by n^{i+1}, so is $r_i + d_i(2n)^i$. But this quantity is then at least n^{i+1} and less than $(2n)^i + d_i(2n)^i$, which yields $d_i > n/2^i - 1$. Therefore, if $n \ge 2^{k-1}(d+1)$, then all digits in the base-$2n$ representation of n^k are greater than d.

Problem 2 $(272; 153, 13, 29, 1, 0, 30, 2, 44)$; *proposed by Carl Schildkraut*. Let n and k be positive integers with $k < n$. Let $P(x)$ be a polynomial of degree n with real coefficients, nonzero constant term, and no repeated roots. Suppose that for any real numbers a_0, a_1, $\dots$, a_k such that the polynomial $a_k x^k + \cdots + a_1 x + a_0$ divides $P(x)$, the product $a_0 a_1 \cdots a_k$ is zero. Prove that $P(x)$ has a nonreal root.

Solution. We will prove the following claim for every positive integer d: If a polynomial of degree d with real coefficients and nonzero constant term has d pairwise distinct real roots, then it has a divisor of degree $d-1$ whose coefficients are all nonzero. The statement to be proven can then be established by assuming indirectly that $P(x)$ has only real roots and applying the claim successively $n-k$ times, which would result in a polynomial divisor of $P(x)$ of degree k with all nonzero coefficients, a contradiction.

To prove our claim, let $p(x)$ be a polynomial of degree d with real coefficients and nonzero constant term, and suppose that it has pairwise distinct real roots $r_1, \dots, r_d$. Since our claim is trivial for $d = 1$, we assume that $d \ge 2$. We may also assume that $p(x)$ is monic, so

$$p(x) = (x - r_1)(x - r_2)\cdots(x - r_d).$$

Consider the d monic divisors of $p(x)$ of degree $d-1$, namely $p_i(x) = p(x)/(x - r_i)$ as $i = 1, \dots, d$. Suppose for the sake of contradiction that, for each i, there is some ℓ_i for which the coefficient of x^{ℓ_i} in $P_i(x)$ is zero. Since $p_i(x)$ is monic and its constant term is not zero, we have $1 \le \ell_i \le d-2$. By the Pigeonhole Principle, we can find some indices $i_1 \ne i_2$ such that $\ell_{i_1} = \ell_{i_2} = \ell$, that is, the coefficients of x^ℓ in $p_{i_1}(x)$ and $p_{i_2}(x)$ are both zero.

Let us now consider $q(x) = p(x)/((x - r_{i_1})(x - r_{i_2}))$, and write

$$q(x) = b_{d-2}x^{d-2} + b_{d-3}x^{d-3} + \cdots + b_1 x + b_0$$

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(with $b_{d-2} = 1$). We then have

$$p_{i_1}(x) = (x - r_{i_1})q(x) = b_{d-2}x^{d-1} + (b_{d-3} - b_{d-2}r_{i_1})x^{d-2} + \cdots + (b_0 - b_1 r_{i_1})x - b_0 r_{i_1},$$

and thus $b_{\ell-1} - b_\ell r_{i_1} = 0$. Similarly, we have $b_{\ell-1} - b_\ell r_{i_2} = 0$. But then $b_{\ell-1} = b_\ell = 0$. Since $p(x)$ and thus $q(x)$ have a nonzero constant term, this also shows that we have $\ell \geq 2$ and thus $d \geq 4$.

Next, we consider the sequence of coefficients in $q(x)$. Let A be the number of sign changes between consecutive nonzero coefficients where the indices (corresponding to the exponents of x) change parity, B be the number of sign changes between consecutive nonzero coefficients where the indices do not change parity, C be the number of consecutive nonzero coefficients of the same sign where the indices change parity, and D be the number of consecutive nonzero coefficients of the same sign where the indices do not change parity. Note that we must have at least one zero coefficient between any two consecutive nonzero coefficients where the indices do not change parity. We know that b_ℓ and $b_{\ell-1}$ are zero; suppose that they are between consecutive nonzero coefficients b_j and b_m. If j and m have opposite parity, then these two zero coefficients have not been counted yet, and if they have the same parity, then we must have at least three zero coefficients between them. In either case, the number of consecutive pairs of indices in our sequence (which equals $d-2$) is at least $A + 2B + C + 2D + 2$.

Recall also that according to Descartes' Rule of Signs, the number of positive real roots of $q(x)$ is bounded above by $A + B$, while the number of negative real roots of $q(x)$ is bounded above by $B + C$ (indeed, consider the positive real roots of $q(-x)$). Given that $q(0) \neq 0$, the number of real roots of $q(x)$ is bounded above by $A + 2B + C$.

Combining our previous two paragraphs, we get

$$d - 2 \leq A + 2B + C \leq A + 2B + C + 2D \leq d - 4,$$

which is a contradiction.

Problem 3 $(185; 154, 4, 0, 4, 0, 1, 4, 18)$; *proposed by Carl Schildkraut.* Alice the architect and Bob the builder play a game. First, Alice chooses two points P and Q in the plane and a subset $\mathcal{S}$ of the plane, which are announced to Bob. Next, Bob marks infinitely many points in the plane, designating each a city. He may not place two cities within distance at most one unit of each other, and no three cities he places may be collinear. Finally, roads are constructed between the cities as follows: for each pair A, B of cities, they are connected with a road along the line segment AB if and only if the following condition holds:

> For every city C distinct from A and B, there exists $R \in \mathcal{S}$ such that $\triangle PQR$ is directly similar to either $\triangle ABC$ or $\triangle BAC$.

Alice wins the game if (i) the resulting roads allow for travel between any pair of cities via a finite sequence of roads and (ii) no two roads cross. Otherwise, Bob wins. Determine, with proof, which player has a winning strategy.

Note: $\triangle UVW$ is *directly similar* to $\triangle XYZ$ if there exists a sequence of rotations, translations, and dilations sending U to X, V to Y, and W to Z.

Solution. We claim that Alice wins the game if, after choosing points P and Q arbitrarily, she sets $\mathcal{S}$ to be the set of points outside the disk of diameter PQ.

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Let $\mathcal{T}$ be Bob's choice for his infinite set of cities. Note that $A \in \mathcal{T}$ and $B \in \mathcal{T}$ will be connected by a line segment if and only if for every city $C \in \mathcal{T}$ different from A and B, $\angle ACB$ is an acute angle. This immediately implies that in the resulting set of roads no two roads will cross: indeed, if segment AB were to cross segment CD for some cities A, B, C, and D, then the quadrilateral $ACBD$ would have four acute angles, which is impossible.

We need to prove that any pair of cities is connected via a finite sequence of roads. We use induction, and prove that any pair of cities that are of distance less than $\sqrt{n}$ for some $n \in \mathbb{N}$ are connected. Since this statement is vacuously true for $n = 1$, we may assume that, for a fixed integer $n \geq 2$, any pair of cities that are less $\sqrt{n-1}$ apart are connected by a finite sequence of roads, and consider cities A and B with distance less than $\sqrt{n}$. If these two cities are not connected directly, then there must be a city $C \in \mathcal{T}$ for which $\angle ACB \geq 90°$. That implies that $AC^2 + CB^2 \leq AB^2$, and since $AC > 1$, $BC > 1$, and $AB < \sqrt{n}$, this yields $AC < \sqrt{n-1}$ and $BC < \sqrt{n-1}$. Therefore, A is connected to C by a finite sequence of roads, and C is connected to B by a finite sequence of roads. Therefore, A and B are connected by a finite sequence of roads, as claimed.

Problem 4 $(294; 59, 8, 1, 1, 4, 0, 25, 196)$; *proposed by Carl Schildkraut.* Let H be the orthocenter of acute triangle ABC, let F be the foot of the altitude from C to AB, and let P be the reflection of H across BC. Suppose that the circumcircle of triangle AFP intersects line BC at two distinct points X and Y. Prove that C is the midpoint of XY.

Solution. We start by proving the (well-known) fact that P is on the circumcircle ω of $\triangle ABC$. Indeed, $\angle BAP = \angle FCB$ since they are both complementary angles of $\angle ABC$. Because P is the reflection of H across BC, BC is the angle bisector of $\angle FCP$. Therefore, $\angle BAP = \angle BCP$, implying that P lies on ω, as claimed.

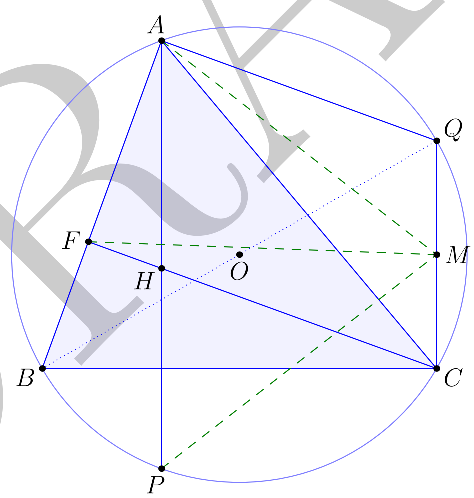

Figure 1: Illustration for Problem 4.

Now let us introduce three additional points: we let O be the center of ω, Q be the antipode of B on ω, and M be the midpoint of QC; see Figure 1.

By Thales's Theorem, QC is perpendicular to BC, and QA is perpendicular to AB. Therefore, AP and QC are parallel, so $AQCP$ is a trapezoid and, since cyclic, it is an isosceles trapezoid. Therefore, $MA = MP$. Furthermore, $AFCQ$ is a right trapezoid and, since M lies on its median

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that is the perpendicular bisector of AF, we get $MA = MF$. But then M is the center of the circumcircle of $\triangle AFP$, so $MX = MY$; since MC is perpendicular to the line BC, this implies that C is the midpoint of XY, as claimed.

Problem 5 $(286; 63, 129, 3, 1, 3, 5, 8, 74)$; *proposed by John Berman.* Determine, with proof, all positive integers k such that

$$\frac{1}{n+1}\sum_{i=0}^{n}\binom{n}{i}^k$$

is an integer for every positive integer n.

Solution. We claim that the property holds if and only if k is even. To see that odd values of k fail, note that for $n = 2$, we have

$$\binom{2}{0}^k + \binom{2}{1}^k + \binom{2}{2}^k = 2 + 2^k,$$

which is congruent to 1 mod 3 when k is odd.

Suppose now that k is even. We establish our claim by proving more generally that

$$S_m = \sum_{i=0}^{n}\binom{n}{\min\{i,m\}}^k = \sum_{i=0}^{m}\binom{n}{i}^k + (n-m)\binom{n}{m}^k$$

is divisible by $n+1$ for every $m = 0, 1, \ldots, n$. The case of $m = n$ gives our result.

We proceed recursively. Since $S_0 = n+1$, our claim holds for $m = 0$. Suppose then that S_m is divisible by $n+1$ for some $0 \le m \le n-1$, and consider S_{m+1}. We observe that

$$\begin{aligned} S_{m+1} - S_m &= \sum_{i=0}^{m+1}\binom{n}{i}^k + (n-m-1)\binom{n}{m+1}^k - \sum_{i=0}^{m}\binom{n}{i}^k - (n-m)\binom{n}{m}^k \\ &= (n-m)\left[\binom{n}{m+1}^k - \binom{n}{m}^k\right]. \end{aligned}$$

Since k is even, $\binom{n}{m+1}^k - \binom{n}{m}^k$ is divisible by $\binom{n}{m+1} + \binom{n}{m}$, and therefore $S_{m+1} - S_m$ is divisible by

$$(n-m)\left[\binom{n}{m+1} + \binom{n}{m}\right] = (n-m)\binom{n+1}{m+1} = (n+1)\binom{n}{m+1}.$$

Thus we see that $S_{m+1} - S_m$ is divisible by $n+1$, and by our assumption so is S_m, so S_{m+1} must be divisible by $n+1$ as well. This completes our proof.

Problem 6 $(188; 168, 10, 1, 0, 2, 0, 0, 7)$; *proposed by Cheng-Yin Chang and Hung-Hsun Yu.* Let m and n be positive integers with $m \ge n$. There are m cupcakes of different flavors arranged around a circle and n people who like cupcakes. Each person assigns a nonnegative real number score to each cupcake, depending on how much they like the cupcake. Suppose that for each person P, it is possible to partition the circle of m cupcakes into n groups of consecutive cupcakes so that the sum of P's scores of the cupcakes in each group is at least 1. Prove that it is possible to distribute the m cupcakes to the n people so that each person P receives cupcakes of total score at least 1 with respect to P.

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Solution. We use strong induction on n. The claim clearly holds for $n = 1$, so suppose that the statement is true for all positive integers less than some $n \geq 2$, and consider n people for whom the problem assumptions hold.

We say that a person is *happy* with a consecutive set of cupcakes if that set has sum at least 1 according to that person. We select one person A, and partition the cupcakes into consecutive sets $C_1, \ldots, C_n$, each of which A is happy with. We refer to $C_1, \ldots, C_n$ as the A-*approved sets*.

We carry out a procedure consisting of several (perhaps zero) rounds, where in each round some of the people get hats, and some of the cupcakes get hidden by a cover. At the beginning, nobody has a hat, and none of the cupcakes are covered. In the first round, if it is possible, a host (not one of the n people) selects a set Ω_1 of people with the property that there are fewer than $|\Omega_1|$ A-approved sets that anyone in Ω_1 is happy with; each person in Ω_1 gets a hat, and each of the A-approved sets that at least one person in Ω_1 was happy with gets covered. If possible, the host continues similarly by selecting a set Ω_2 of people that do not have a hat yet, so that there are fewer than $|\Omega_2|$ A-approved sets among the uncovered A-approved sets that anyone in Ω_2 is happy with; people in Ω_2 get hats, and their A-approved sets get covered. The rounds continue until they are no longer possible, that is, for any set Ω of people with no hats, the number of uncovered A-approved sets that at least one person in Ω is happy with is at least $|\Omega|$. Then, according to Hall's Marriage Theorem, each person without a hat can be matched with a different uncovered A-approved set. The people without hats then leave happily with their assigned A-approved sets of cupcakes. Note that A is happy with every A-approved set and thus never gets a hat, so A will be one of the people who departed.

We are then left with some $k < n$ people, all with hats, and k A-approved sets of cupcakes. If $k = 0$, we are done, so assume that $k \geq 1$. The host now uncovers all the remaining cupcakes (and perhaps slides them closer to each other to make pairs of cupcakes consecutive when there are no remaining cupcakes between them). We claim that, for each remaining person, these remaining cupcakes can be partitioned into k groups of consecutive cupcakes, each of total score at least 1. If we prove this, we will have reduced the problem to the induction hypothesis, so we will be done.

Let B be one of the people remaining who received a hat earlier, and suppose that B partitioned the originally available m cupcakes into consecutive sets $D_1, \ldots, D_n$, each of which B was happy with. Call $D_1, \ldots, D_n$ the B-*approved sets*. When B was assigned a hat, every A-approved set that B would be happy with became covered. Therefore, B is not happy with any of the A-approved sets that were already given away, which is to say that B assigned a score less than 1 to each of these sets. In particular, each of the A-approved sets that was already given away intersects at most two B-approved sets. Among these A-approved sets that were already given away, let $E_1, \ldots, E_p$ be those that intersect two B-approved sets, and $F_1, \ldots, F_q$ be those that intersect exactly one B-approved set. Since $n - k$ A-approved sets were given away, $p + q = n - k$.

Now apply the following procedure (see Figure 2 for an illustration). First, combine the two B-approved sets that intersect at E_1, the two that intersect at E_2, and so on to E_p. Now we have $n - p$ sets of consecutive cupcakes. Among these, cover those that intersect any of $F_1, \ldots, F_q$. Now we have at least $n - p - q = n - (n - k) = k$ sets of consecutive uncovered cupcakes. Any such set X consists of some number $r \geq 1$ of B-approved sets, but missing $r - 1$ A-approved sets that have already been given away. Since B assigned a value of at least 1 to each B-approved set but a value less than 1 to each A-approved set that was already given away, the total value of X for B is at least $r - (r - 1) = 1$. Therefore, B has at least k sets of consecutive uncovered cupcakes that B is happy with. Since our selection of B was arbitrary, this is true for any remaining person, so

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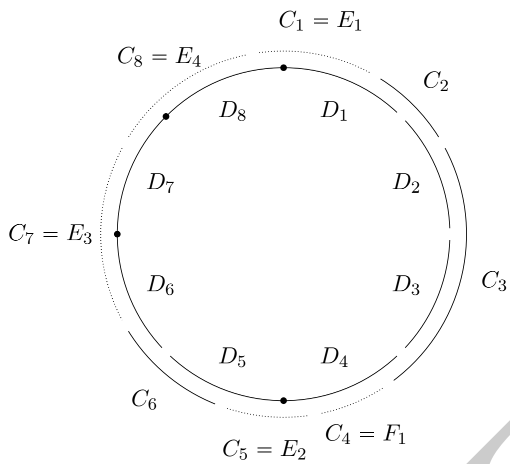

Figure 2: Illustration for Problem 6.

the induction hypothesis applies, which is what we had to prove.

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Summary. We present the problems and solutions to the 54th Annual United States of America Mathematical Olympiad.

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